**Recurrence Relations**

(A continuation from finding the rule for the nth term)

|  |  |
| --- | --- |
| Term to Term‘Open form equation’ | Position to Term‘Closed form equation’ |
| ‘Start at two, then add three each time’ | Multiply the position by three & subtract one |
| $U\_{n+1}=U\_{n}+3$, $U\_{1}=2$ | $$U\_{n}=3n-1$$ |
| 2, 5, 8… | 2, 5, 8… |

**Definitions**

|  |  |  |
| --- | --- | --- |
|  | First Order | Second Order |
| Homogeneous | $$U\_{n}=3U\_{n-1}$$ | $$U\_{n}=3U\_{n-1}+7U\_{n-2}$$ |
| Non - homogeneous | $$U\_{n}=3U\_{n-1}+k$$$$U\_{n}=3U\_{n-1}+3n^{2}$$ | $$U\_{n}=3U\_{n-1}+7U\_{n-2}+k$$$$U\_{n}=+7U\_{n-2}+3n^{2}$$ |

A **linear recurrence relation** is one where the next term is a multiple of the previous term, with no squaring, cubing etc.

**Useful Equations for Solving Non-Homogeneous Equations**

|  |  |
| --- | --- |
| Form of Non-Homogeneous Part | Useful Equation |
| $c$ (i.e. a constant) | $$a$$ |
| $cn$ (i.e. a linear multiple of $n)$  | $$an+b$$ |
| $$cn^{2}$$ | $$an^{2}+bn+d$$ |
| $$cr^{n}$$ | $$ar^{n}$$ |
| $$cn^{2}r^{n}$$ | $$r^{n}\left(an^{2}+bn+d\right)$$ |

**Example 1** – find the closed form equation of a first order, linear, homogeneous equation

$U\_{n}=4U\_{n-1}$, $U\_{1}=2$

The next term is the previous one multiplied by four and the first term is two.

i.e.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Position | $$1$$ | $$2$$ | $$3$$ | $$4$$ | $$n$$ |
| Term | $$2$$ | $$2×4$$ | $$2×4^{2}$$ | $$2×4^{3}$$ | $$…2×4^{n-1}$$ |

**Note…**

In the absence of the first term, we would know the format of the sequence is $4^{n}$ but wouldn’t know the exact sequence for sure. We could find a general solution but need the starting term to obtain a particular solution.

**Example 2** – find the closed form equation of a first order, linear, homogeneous equation

$$U\_{n}=5U\_{n-1}$$

The next term is the previous one multiplied by five.

i.e…

$U\_{n}=5^{n-1}$ 🡨 this is known as the auxiliary equation

Given $U\_{1}=3$…

$$U\_{n}=3×5^{n-1}$$

**Note…**

Equivalently, staring with a first term $U\_{0}$ (where it’s position one but $n=0$), this would be…

$U\_{n}=5U\_{n-1}$, $U\_{0}=3$

$$U\_{n}=3×5^{n}$$

**Example 1 again** – a better general method for finding the closed form equation of a first order, linear, homogeneous equation, that we’ll also use later when dealing with non-homogeneous equations

$U\_{n}=4U\_{n-1}$, $U\_{0}=2$

Er, what power?, try $U\_{n}=x^{n}$, where also $U\_{n-1}=x^{n-1}$…

$$U\_{n}=4U\_{n-1}$$

$$⟹x^{n}=4x^{n-1}$$

$$⟹x^{n}=\frac{4x^{n}}{x}$$

$⟹x=4$ 🡨 this is the auxiliary equation

$$⟹U\_{n}=4^{n}$$

$U\_{n}=k4^{n}$ 🡨 this is the general solution

$$U\_{0}=2⟹U\_{0}=2=k4^{0}$$

$$⟹k=2$$

$⟹U\_{n}=2×4^{n}$ 🡨 this is the particular solution

**Example 3** – find the closed form equation of a first order, linear, non-homogeneous equation

$U\_{n}=3U\_{n-1}+4$, $U\_{0}=2$

We’ll solve the homogeneous part of the equation first, then separately deal with the +4, then put both parts together.

$x=3$ (auxiliary equation)

$U\_{n}=k3^{n}$ 🡨 non-homogeneous equation so this is known as the complementary function

The non-homogeneous part is a constant so use $a$ as the useful equation.

$U\_{n}=a$ and $U\_{n-1}=a$

$$U\_{n}=3U\_{n-1}+4$$

$$⟹a=3a+4$$

$$⟹a=-2$$

Putting both parts together gives…

$U\_{n}=k3^{n}-2$ (general solution)

Initial condition…

$U\_{0}=2$ (and $n=0$)

$$⟹ 2=k3^{0}-2$$

$$⟹ k=4$$

So the particular solution is…

$$U\_{n}=4×3^{n}-2$$

**Example 4** - find the closed form equation of a first order, linear, non-constant non-homogeneous equation

$U\_{n}=6U\_{n-1}+3^{n}$, $U\_{0}=4$

We’ll solve the homogeneous part of the equation first, then separately deal with the $3^{n}$, then put both parts together.

$x=6$ (auxiliary equation)

$U\_{n}=k6^{n}$ (complementary function)

The non-homogeneous part is $3^{n}$so use $ar^{n}$ as the useful equation, where $r=3$.

$U\_{n}=a3^{n}$ and $U\_{n-1}=a3^{n-1}$

$$⟹a3^{n}=6a3^{n-1}+3^{n}$$

$$⟹a3^{n}=\frac{6a3^{n}}{3}+3^{n}$$

$$⟹a3^{n}=2a3^{n}+3^{n}$$

$$⟹a=-1$$

Putting both parts together gives…

$U\_{n}=k6^{n}-1×3^{n}$ (general solution)

Initial condition…

$U\_{0}=4$ (and $n=0$)

$$⟹4=k6^{0}-1×3^{0}$$

$$⟹ k=5$$

So the particular solution is…

$$U\_{n}=5×6^{n}-3^{n}$$

Your turn…