Recurrence Relations

(A continuation from finding the rule for the nth term)

Term to Term 'Open form equation'	Position to Term 'Closed form equation'
'Start at two, then add three each time'	Multiply the position by three & subtract one
$U_{n+1} = U_n + 3, \;\; U_1 = 2$	$U_n = 3n - 1$
2, 5, 8…	2, 5, 8…

Definitions

	First Order	Second Order
Homogeneous	$U_n = 3U_{n-1}$	$U_n = 3U_{n-1} + 7U_{n-2}$
Non - homogeneous	$U_n = 3U_{n-1} + k$ $U_n = 3U_{n-1} + 3n^2$	$U_n = 3U_{n-1} + 7U_{n-2} + k$ $U_n = +7U_{n-2} + 3n^2$

A **linear recurrence relation** is one where the next term is a multiple of the previous term, with no squaring, cubing etc.

Useful Equations for Solving Non-Homogeneous Equations

Form of Non-Homogeneous Part	Useful Equation
c (i.e. a constant)	a
cn (i.e. a linear multiple of n)	$an + b$
cn^2	$an^2 + bn + d$
cr^n	ar^n
cn^2r^n	$r^n(an^2 + bn + d)$

Example 1 – find the closed form equation of a first order, linear, homogeneous equation

$$U_n = 4U_{n-1}, \quad U_1 = 2$$

The next term is the previous one multiplied by four and the first term is two.

i.e.

Position	1	2	3	4	n
Term	2	2×4	2×4^2	2×4^3	… $2 \times 4^{n-1}$

Note…

In the absence of the first term, we would know the format of the sequence is 4^n but wouldn't know the exact sequence for sure. We could find a general solution but need the starting term to obtain a particular solution.

Example 2 – find the closed form equation of a first order, linear, homogeneous equation

$$U_n = 5U_{n-1}$$

The next term is the previous one multiplied by five.

i.e…

$$U_n = 5^{n-1}$$ ← this is known as the auxiliary equation

Given $U_1 = 3$…

$$U_n = 3 \times 5^{n-1}$$

Note…

Equivalently, staring with a first term U_0 (where it's position one but $n = 0$), this would be…

$$U_n = 5U_{n-1}, \quad U_0 = 3$$

$$U_n = 3 \times 5^n$$

Example 1 again – a better general method for finding the closed form equation of a first order, linear, homogeneous equation, that we'll also use later when dealing with non-homogeneous equations

$$U_n = 4U_{n-1}, \quad U_0 = 2$$

Er, what power?, try $U_n = x^n$, where also $U_{n-1} = x^{n-1}$…

$$U_n = 4U_{n-1}$$

$$\Rightarrow x^n = 4x^{n-1}$$

$$\Rightarrow x^n = \frac{4x^n}{x}$$

$$\Rightarrow x = 4$$ ← this is the auxiliary equation

$$\Rightarrow U_n = 4^n$$

$$U_n = k4^n$$ ← this is the general solution

$$U_0 = 2 \Rightarrow U_0 = 2 = k4^0$$

$$\Rightarrow k = 2$$

$$\Rightarrow U_n = 2 \times 4^n$$ ← this is the particular solution

Example 3 – find the closed form equation of a first order, linear, non-homogeneous equation

$$U_n = 3U_{n-1} + 4, \quad U_0 = 2$$

We'll solve the homogeneous part of the equation first, then separately deal with the +4, then put both parts together.

$$x = 3 \quad \text{(auxiliary equation)}$$

$U_n = k3^n$ ← non-homogeneous equation so this is known as the complementary function

The non-homogeneous part is a constant so use a as the useful equation.

$$U_n = a \quad \text{and} \quad U_{n-1} = a$$

$$U_n = 3U_{n-1} + 4$$

$$\Rightarrow a = 3a + 4$$

$$\Rightarrow a = -2$$

Putting both parts together gives…

$$U_n = k3^n - 2 \quad \text{(general solution)}$$

Initial condition…

$$U_0 = 2 \quad (\text{and } n = 0)$$

$$\Rightarrow 2 = k3^0 - 2$$

$$\Rightarrow k = 4$$

So the particular solution is…

$$U_n = 4 \times 3^n - 2$$

Example 4 - find the closed form equation of a first order, linear, non-constant non-homogeneous equation

$$U_n = 6U_{n-1} + 3^n, \quad U_0 = 4$$

We'll solve the homogeneous part of the equation first, then separately deal with the 3^n, then put both parts together.

$$x = 6 \quad \text{(auxiliary equation)}$$

$$U_n = k6^n \quad \text{(complementary function)}$$

The non-homogeneous part is 3^nso use ar^n as the useful equation, where $r = 3$.

$$U_n = a3^n \quad \text{and} \quad U_{n-1} = a3^{n-1}$$

$$\Rightarrow a3^n = 6a3^{n-1} + 3^n$$

$$\Rightarrow a3^n = \frac{6a3^n}{3} + 3^n$$

$$\Rightarrow a3^n = 2a3^n + 3^n$$

$$\Rightarrow a = -1$$

Putting both parts together gives...

$$U_n = k6^n - 1 \times 3^n \quad \text{(general solution)}$$

Initial condition...

$$U_0 = 4 \quad (\text{and } n = 0)$$

$$\Rightarrow 4 = k6^0 - 1 \times 3^0$$

$$\Rightarrow k = 5$$

So the particular solution is...

$$U_n = 5 \times 6^n - 3^n$$

Your turn...