AQA Core 4 Vectors – Probing Questions

Write on the board the vector positions (coordinates) of three points in 3D space. Ask students for sensible random values for the first point or use our values of $A = (3,7,1)$;

$$A = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad C = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$$

Using mini-whiteboards, ask students for:

1. The vector $\overrightarrow{AB}$
2. The distance $\overrightarrow{AB}$
3. The type of triangle (scalene, isosceles, equilateral)
4. Whether the triangle ABC is right-angled or not

Most likely the students will have chosen coordinates for A that do not give a right-angled triangle this raises the questions;

5. Find the scalar product of B and C and comment on the significance of this. Compare this answer with your answer for (4).

6. a) How can we change/fix the question to make ABC a right-angled triangle?
   b) Can you suggest three coordinates that do form a right-angled triangle in 3D space?

If students have already formed a right-angled triangle then rephrase the question such as "What would we need to do to ensure that the triangle is not right-angled?"

7. Having found methods for finding right-angled triangles in 3D space, how can we form isosceles, right-angled triangles?

8. Reverting to our original coordinates, what is the shortest distance from point A to the line joining points B and C, and what are the coordinates of this point?

9. Compare and comment on the relationships of the angles between:

   i) $\overrightarrow{AB}$ and $\overrightarrow{BC}$ vs $\overrightarrow{BA}$ and $\overrightarrow{BC}$
   ii) $\overrightarrow{AB}$ and $\overrightarrow{CB}$ vs $\overrightarrow{BA}$ and $\overrightarrow{CB}$

The following is an extension question typically for students studying Further Mathematics;

10. What is the vector equation of the plane in which all three of these coordinates lie?

Answers

Question 1 is given by finding the vector $\overrightarrow{AB}$;

$$\overrightarrow{AB} = B - A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -5 \\ 2 \end{pmatrix}$$

Question 2 is given by using Pythagoras on $\overrightarrow{AB}$;

$$\left|\overrightarrow{AB}\right| = \sqrt{(-2)^2 + (-5)^2 + 2^2} = \sqrt{4+25+4} = \sqrt{33}$$

Question 3 is given by using Pythagoras on $\overrightarrow{AC}$ and $\overrightarrow{BC}$ and comparing all three lengths;

$$\overrightarrow{AC} = C - A = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \\ -3 \end{pmatrix}$$

$$\begin{aligned} |C - A| &= \sqrt{1^2 + (-6)^2 + (-3)^2} \\ &= \sqrt{1+36+9} = \sqrt{46} \end{aligned}$$

$$\overrightarrow{BC} = C - B = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -5 \end{pmatrix}$$

$$\begin{aligned} \overrightarrow{BC} &= \sqrt{3^2 + (-1)^2 + (-5)^2} \\ &= \sqrt{9+1+25} = \sqrt{35} \end{aligned}$$

$$\left|\overrightarrow{AB}\right| \neq \left|\overrightarrow{AC}\right| \neq \left|\overrightarrow{BC}\right| \Rightarrow \text{scalene triangle}$$

Question 4 can be answered by finding the scalar product $\overrightarrow{AB} \bullet \overrightarrow{BC}$, $\overrightarrow{AB} \bullet \overrightarrow{AC}$, $\overrightarrow{AC} \bullet \overrightarrow{BC}$ or by checking Pythagoras on the lengths found in question 3;

$$\begin{aligned} \overrightarrow{AB} \bullet \overrightarrow{BC} &= (-2\times 3) + (-5\times -1) + (2\times -5) = -6+5-10 = -11 \\ \overrightarrow{AB} \bullet \overrightarrow{AC} &= (-2\times 1) + (-5\times -6) + (2\times -3) = -2+30-6 = 22 \\ \overrightarrow{AC} \bullet \overrightarrow{BC} &= (1\times 3) + (-6\times -1) + (-3\times -5) = 3+6+15 = 24 \end{aligned}$$

Since none of these scalar products are zero we conclude that the triangle is non-right-angled.

By Pythagoras;

$$33+35 \neq 46 \Rightarrow \text{non-right-angled}$$

Question 5, scalar product of $B \bullet C$ is given by;

$$B \bullet C = (1\times 4) + (2\times 1) + (3\times -2) = 4+2-6 = 0$$

This is equal to zero and therefore suggests a right-angled triangle. However, since B and C are *position vectors* relative to the origin the right angled that we have found is not between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ but instead at the origin and between the *direction vectors* $\overrightarrow{OB}$ and $\overrightarrow{OC}$. Our conclusion that triangle ABC is not right-angled remains valid.

The change required in question 6(a), to produce a right-angled triangle, can be made by using the *position vectors* of B and C as *direction vectors* to form *vector equations of lines* from A.

$$\underline{b} = \begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3+\lambda \\ 7+2\lambda \\ 1+3\lambda \end{pmatrix} \qquad \underline{c} = \begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 3+4\mu \\ 7+\mu \\ 1-2\mu \end{pmatrix}$$

We can then create two new points B', on $\underline{b}$, and C', on $\underline{c}$. Because the direction vectors of $\underline{b}$, and $\underline{c}$ have a scalar product of zero we know that the triangle formed by the points $AB'C'$ is right-angled (with the right-angle at A).

In question 6(b) we are asked to find sets of coordinates in 3D space that form right-angled triangles. We use a similar method to the above. The key to this solution is to begin by giving two direction vectors consisting of three products that sum to zero, such as;

$$\underline{p} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \qquad \underline{q} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

Where:

$$\underline{p} \bullet \underline{q} = (3\times 2) + (4\times -1) + (2\times -1) = 6-4-2 = 0$$

(Note that this is made particularly easy by choosing at least one value to be 1.)

As before, we now use both of these as direction vectors from a common point to form vector equations of lines from this point. We have chosen our common point as $D = (7,4,2)$ and will label the other points E and F

$$\overrightarrow{DE} = \begin{pmatrix} 7 \\ 4 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 7+3\lambda \\ 4+4\lambda \\ 2+2\lambda \end{pmatrix} \qquad \overrightarrow{DF} = \begin{pmatrix} 7 \\ 4 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 7+2\mu \\ 4-\mu \\ 2-\mu \end{pmatrix}$$

What is the significance of the point D? Could we have used any other coordinates? What restrictions must the point D fulfil?

Question 7, to create the isosceles right-angled triangle. Having established how to create right-angled triangles in 3D space by finding points on perpendicular vector equations of lines from a common point, we now need to place restrictions on the lengths of these lines to ensure that they are the same length. We'll use our triangle DEF as an example;

If we choose $\lambda = 1$ then;

$$\left|\overrightarrow{DE}\right| = \sqrt{(7+3)^2+(4+4)^2+(2+2)^2} = \sqrt{10^2+8^2+4^2} = \sqrt{100+64+16} = \sqrt{180} = 6\sqrt{5}$$

We now find a value of μ so that $\left|\overrightarrow{DF}\right| = 6\sqrt{5}$;

$$\sqrt{(7+2\mu)^2+(4-\mu)^2+(2-\mu)^2} = 6\sqrt{5}$$
$$(7+2\mu)^2+(4-\mu)^2+(2-\mu)^2 = 180$$
$$49+4\mu+4\mu^2+16-8\mu+\mu^2+4-4\mu+\mu^2 = 180$$
$$6\mu^2-8\mu+69 = 180$$
$$6\mu^2-8\mu-11 = 0$$

By quadratic formula solutions are;

$$\mu = \frac{4 \pm \sqrt{82}}{6}$$

And therefore, coordinates of point F are;

$$F = \begin{pmatrix} 7 \\ 4 \\ 2 \end{pmatrix} + \frac{4 \pm \sqrt{82}}{6} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Why are there two possible values of μ?

In question 8, to find the shortest distance from point A to the line joining points B and C, the process is;

1. Find the vector equation of the line through $\overrightarrow{BC}$
2. Find the coordinates, in terms of λ, of the point X on $\overrightarrow{BC}$ that is closest to A
3. Find the vector $\overrightarrow{XA}$ in terms of λ
4. Find the value of λ by using the scalar product of the vector $\overrightarrow{XA}$ and direction vector of $\overrightarrow{BC}$ equals zero.
5. Substitute the value of λ into X to get actual coordinates for X.
6. Substitute the value of λ into $\overrightarrow{XA}$ to get actual vector $\overrightarrow{XA}$.
7. Find $\left|\overrightarrow{XA}\right|$.

1) Vector equation of the line $\overrightarrow{BC}$. One equation of this line is;

$$\underline{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ -5 \end{pmatrix}$$

2) Point on $\overrightarrow{BC}$, X, that is closest to our point A;

$$X = \begin{pmatrix} 1+3\lambda \\ 2-\lambda \\ 3-5\lambda \end{pmatrix}$$

3) The vector $\overrightarrow{XA}$;

$$\overrightarrow{XA} = \begin{pmatrix} 1+3\lambda \\ 2-\lambda \\ 3-5\lambda \end{pmatrix} - \begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix} = \begin{pmatrix} -2+3\lambda \\ -5-\lambda \\ 2-5\lambda \end{pmatrix}$$

4) At the point X the scalar product of the line $\overrightarrow{XA}$ and the direction vector of the line $\overrightarrow{BC}$ will be zero;

$$\overrightarrow{XA} \bullet \begin{pmatrix} 3 \\ -1 \\ -5 \end{pmatrix} = 3(-2+3\lambda) - (-5-\lambda) - 5(2-5\lambda) = 35\lambda - 11 = 0 \Rightarrow \lambda = \frac{11}{35}$$

5) Actual coordinates for X;

$$X = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \frac{11}{35} \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{68}{35} \\ \frac{59}{35} \\ \frac{10}{7} \end{pmatrix}$$ (the coordinates of the closest point on the line to A)

6) Actual vector $\overrightarrow{XA}$;

$$\overrightarrow{XA} = \begin{pmatrix} -2+3\left(\frac{11}{35}\right) \\ -5-\left(\frac{11}{35}\right) \\ 2-5\left(\frac{11}{35}\right) \end{pmatrix} = \begin{pmatrix} \frac{-37}{35} \\ \frac{-208}{35} \\ \frac{3}{7} \end{pmatrix}$$

7) Size of $\overrightarrow{XA}$;

$$\left|\overrightarrow{XA}\right| = \sqrt{\left(\tfrac{-37}{35}\right)^2 + \left(\tfrac{-208}{35}\right)^2 + \left(\tfrac{3}{7}\right)^2} = \sqrt{\tfrac{1369}{1225} + \tfrac{43264}{1225} + \tfrac{9}{49}} = \sqrt{\tfrac{44858}{1225}} \approx 6.05$$