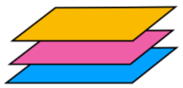
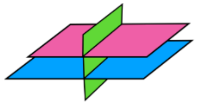
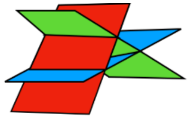
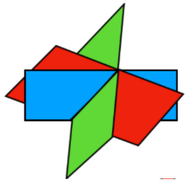
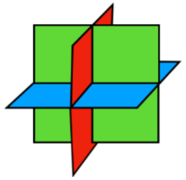
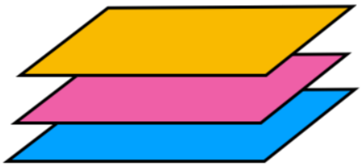
Intersection of Three Planes

Type	Image	Example	Normals Parallel?	Coefficient Multiples?	Normals Coplanar?	$Det(M) =$	Planes Consistent?	Number of Solutions	Other Notes
All Parallel		$6x - 15y - 3z + 9 = 0$ $2x - 5y - z - 1 = 0$ $4x - 10y - 2z - 4 = 0$	All normals parallel	Coefficients are multiples, (constants not multiples)		$Det(M) = 0$ $n_1 \cdot n_2 \times n_3 = 0$	Planes are inconsistent	No solutions	
Two Parallel and the other not		$6x - 15y - 3z + 9 = 0$ $2x - 5y - z - 1 = 0$ $5x - 10y - 3z - 4 = 0$	Only two normals parallel	Only two sets of coefficients are multiples		$Det(M) = 0$ $n_1 \cdot n_2 \times n_3 = 0$	Planes are inconsistent	No solutions	
Triangular Prism		$6x - 5y + 3z - 1 = 0$ $4x + y - z + 5 = 0$ $x - 3y + 2z + 7 = 0$	No normals parallel	No coefficient multiples	All three normals coplanar	$Det(M) = 0$ $n_1 \cdot n_2 \times n_3 = 0$	Planes are inconsistent	No solutions	One normal is a combination of multiples of the other normals
A Sheaf		$6x - 5y + 3z + 19 = 0$ $4x + y - z + 5 = 0$ $x - 3y + 2z + 7 = 0$	No normals parallel	No coefficient multiples	All three normals coplanar	$Det(M) = 0$ $n_1 \cdot n_2 \times n_3 = 0$	Planes are consistent	Infinite solutions, equation of a line	One full equation is a combination of multiples of the other two
Single Point		$6x - 5y + 3z + 19 = 0$ $4x + y - z + 5 = 0$ $x - 3y + 2z + 7 = 0$	No normals parallel	No coefficient multiples	Normals not coplanar	$Det(M) \neq 0$ $n_1 \cdot n_2 \times n_3 \neq 0$	Planes are consistent	One unique solution	

Intersection of Three Planes

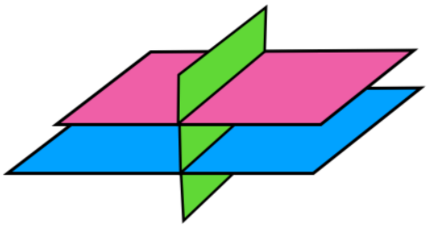
All Parallel



$$6x - 15y - 3z + 9 = 0$$
$$2x - 5y - z - 1 = 0$$
$$4x - 10y - 2z - 4 = 0$$

- All normals parallel so are multiples of each other. In example above, the blue numbers are all multiples.
- The constants are not in the same multiples as, if they were, the equations would represent the same plane
- $Det(M) = 0$
- Planes are inconsistent since no points on all three planes, i.e no solutions

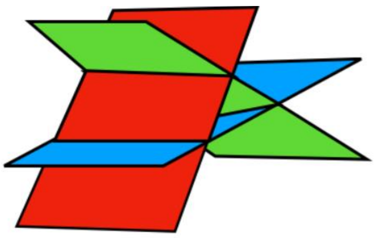
Two Parallel (and the other not)



$$6x - 15y - 3z + 9 = 0$$
$$2x - 5y - z - 1 = 0$$
$$5x - 10y - 3z - 4 = 0$$

- Two normals parallel so are multiples of each other. In example above, the blue numbers are all multiples but the greens aren't.
- The third plane crosses the others along two parallel lines
- $Det(M) = 0$
- Planes are inconsistent since no points on all three planes, i.e no solutions

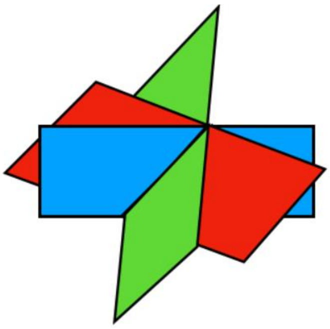
Triangular Prism



$$6x - 5y + 3z - 1 = 0$$
$$4x + y - z + 5 = 0$$
$$x - 3y + 2z + 7 = 0$$

- Planes intersecting in pairs of parallel lines
- No normals parallel so no multiples of each other. In example above, the blues, greens and oranges aren't multiples of each other.
- One of the normals is a combination of multiples of the other two normals. In example above,
  $$\text{blue normal} \;=\; 2 \times \text{green normal} + \text{orange normal}$$
- All three normals are coplanar, i.e they all lie in the same plane.
- $Det(M) = 0$
- Planes are inconsistent since no points on all three planes, i.e no solutions

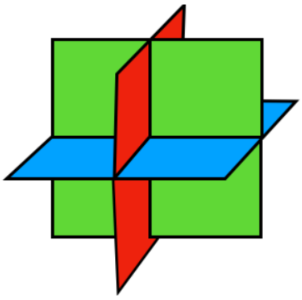
A Sheaf



$$6x - 5y + 3z + 19 = 0$$
$$4x + y - z + 5 = 0$$
$$x - 3y + 2z + 7 = 0$$

- All three planes intersecting in a line
- Similar to triangular prism but with the prism reduced to a single line
- No normals parallel so no multiples of each other. In example above, the blues, greens and oranges aren't multiples of each other.
- One full equation is a combination of multiples of the other two. In example above,
$$\text{blue} = 2 \times \text{green} + \text{orange}$$
- As with the triangular prism, all three normals are coplanar, i.e they all lie in the same plane.
- $Det(M) = 0$
- Planes are consistent since there are points that lie on all three planes.
- The solution is the equation of a line and there are infinite solutions along this line

Single Point



$$6x - 5y + 3z + 19 = 0$$
$$4x + y - z + 5 = 0$$
$$x - 3y + 2z + 7 = 0$$

- All three planes intersect at a single point so there is one specific solution
- Planes are consistent since there is a point that lies on all three planes.
- No normals parallel so no multiples of each other. In example above, the blues, greens and oranges aren't multiples of each other.
- The normals aren't coplanar
- The triple scalar product isn't zero, i.e. $n_1 \cdot (n_2 \times n_3) \neq 0$. This is because the triple scalar product represents the volume of the 3D object formed by these three vectors. If this isn't zero then the normals aren't coplanar and this volume exists.
- $Det(M) \neq 0$

https://www.youtube.com/watch?v=duFRYId7kNU&t=8s

https://www.youtube.com/watch?v=zga9_x-rh3E

https://www.youtube.com/watch?v=V9Lo9M7gwCg