Lots of Proof Questions

Algebraic Proof

Prove algebraically that $n^3 + 3n - 1$ is odd for all positive integers n. [4]
N is an integer that is not divisible by 3. Prove that N^2 is of the form $3p + 1$, where p is an integer. [5]
Prove that the sum of the squares of any two consecutive integers is of the form $4k + 1$, where k is an integer. [4]
By considering separately the case when n is odd and the case when n is even, prove that the following statement is true. n is a positive integer $\Rightarrow n^2 + 1$ is not a multiple of 4. [4]
Prove that the sum of the squares of any three consecutive positive integers cannot be divided by 3. [2]
Tom Cruise claims that "n is an **even** positive integer greater than 2 $\Rightarrow 2^n - 1$ is **not** prime". Prove that Tom's claim is true. [4]

Counter example and Contradiction

Johnny claims that "If n is any positive integer, then $3^n + 2$ is a prime number." Prove that Johnny's claim is incorrect. [3] Amber says that $x = 3 \Leftrightarrow x^2 = 9$. Explain why Amber's statement is incorrect and write a corrected version of Amber's statement. [2]

Prove that the following statement is **not** true. m is an odd number greater than 1 $\Rightarrow m^2 + 4$ is prime. [1]
It is given that n is an integer. Prove by contradiction that n^2 is even $\Rightarrow n$ is even. [5]
A student suggests that, for any prime number between 20 and 40, when its digits are squared and then added, the sum is an odd number. For example, 23 has digits 2 and 3 which gives $2^2 + 3^2 = 13$, which is odd. Show by counter example that this suggestion is false. [2]
Prove by contradiction that $\sqrt{7}$ is irrational. [5]

Exhaustion

Prove by exhaustion that if the sum of the digits of a 2-digit number is 5, then this 2-digit number is not a perfect square. [3]

Others

Show that, if n is a positive integer, then $(x^n - 1)$ is divisible by $(x - 1)$. [1] Hence show that, if k is a positive integer, then $2^{8k} - 1$ is divisible by 17. [4]
Determine the set of values of n for which $\frac{n^2-1}{2}$ and $\frac{n^2+1}{2}$ are positive integers. [3] A 'Pythagorean triple' is a set of three positive integers a, b and c such that $a^2 + b^2 = c^2$. Prove that, for the set of values of n found in part **(a)**, the numbers n, $\frac{n^2-1}{2}$ and $\frac{n^2+1}{2}$ form a Pythagorean triple. [2]