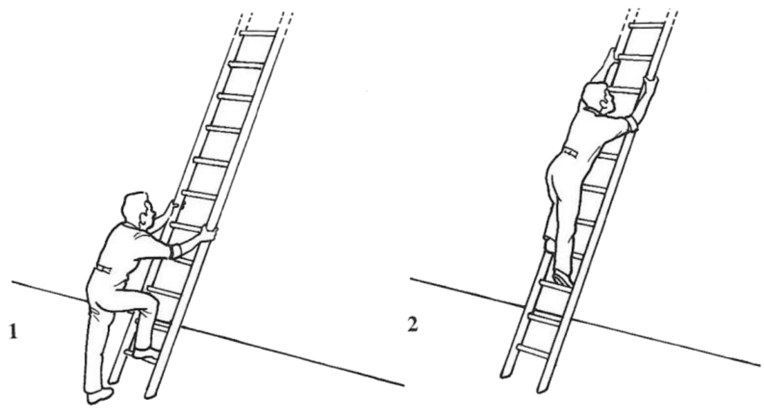
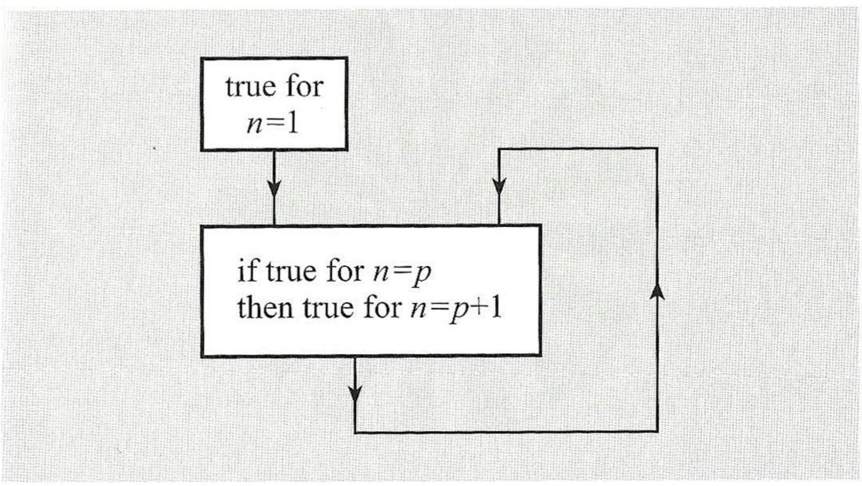
Proof by Induction



	Climbing the ladder	In mathematics
1	Prove that you can reach the bottom rung of the ladder.	Prove that statement is true for $n = 1$.
2	Prove that, from any rung on the ladder that you may be on, you can reach the next rung of the ladder.	Prove that for any value of n, such as k, for which the statement is true, then it will also be true for $k + 1$.
3	State that you have demonstrated that you can climb up the ladder ad infintum.	Conclude the argument.



Two More Examples

A sequence is defined by $u_{n+1} = 4u_n - 3$, $u_1 = 2$.

Prove that $u_n = 4^{n-1} + 1$.

SOLUTION

Step 1 For $n = 1$, $u_1 = 4^0 + 1 = 1 + 1 = 2$, so the result is true for $n = 1$.

Step 2 Assume that the result is true for $n = k$, so that $u_k = 4^{k-1} + 1$.
We want to prove that it is true for $n = k + 1$, i.e. that $u_{k+1} = 4u^k + 1$.

$$\begin{aligned} \text{For } n = k + 1,\ u_{k+1} &= 4u_k - 3 \\ &= 4(4^{k-1} + 1) - 3 \\ &= 4 \times 4^{k-1} + 4 - 3 \\ &= 4^k + 1 \end{aligned}$$

Step 3 So if the result is true for $n = k$, then it is true for $n = k + 1$.
Since it is true for $n = 1$, then it is true for all $n \geqslant 1$ by induction.

Prove that $u_n = 4^n + 6n - 1$ is divisible by 9 for all $n \geqslant 1$.

SOLUTION

Step 1 For $n = 1$, $u_1 = 4 + 6 - 1 = 9$, so it is true when $n = 1$.

Step 2 We want to show that

$$u_k \text{ is divisible by } 9 \Rightarrow u_{k+1} \text{ is divisible by } 9.$$

$$\begin{aligned} \text{Now } u_{k+1} &= 4^{k+1} + 6(k + 1) - 1 \\ &= 4 \times 4^k + 6k + 5 \\ &= 4(u_k - 6k + 1) + 6k + 5 \\ &= 4u_k - 18k + 9 \\ &= 4u_k - 9(2k - 1) \end{aligned}$$

Substituting $4^k = u_k - 6k + 1$.

Step 3 Therefore if u_k is a multiple of 9 then so is u_{k+1}.
Since u_1 is a multiple of 9, u_n is a multiple of 9 for all $n \geqslant 1$.

Proof by Induction

This form of proof can be compared with the process of climbing a ladder: if we can

1 reach the bottom rung

and

2 get from one rung to the next,

then we can climb as far as we like up the ladder (figure 5.1).

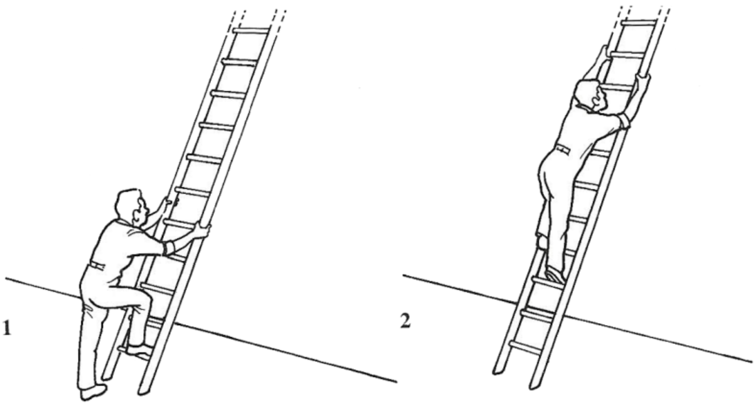


Figure 5.1

The corresponding steps in the proof are

1 showing that the conjecture is true for $n = 1$ (though in fact you checked it up to $n = 6$)

2 showing that if the conjecture is true for any particular value of n, $n = k$ say, then it is true for the next value, $n = k + 1$.

This method of proof is called *proof by mathematical induction* (or just *proof by induction*).

The method of proof by induction can be summarised as follows.

To prove a result by induction you must take three steps.

Step 1 Prove that it is true for a starting value, such as $n = 1$.
Step 2 Prove that if it is true when $n = k$, then it is true when $n = k + 1$.
Step 3 Conclude the argument.

Step 1 is usually a simple verification whereas Step 2 can be quite complicated, so there is a danger that you will concentrate on Step 2 and forget about Step 1 – but it is no use being able to climb the ladder if you cannot reach the bottom rung!

Sums of Squares

Prove that, for all positive integers n,

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

SOLUTION

Step 1 When $n = 1$, L.H.S. $= 1^2 = 1$

R.H.S. $= \tfrac{1}{6} \times 1 \times 2 \times 3 = 1.$

Step 2 Assume that the result is true when $n = k$, so that

$$1^2 + 2^2 + 3^2 + \ldots + k^2 = \tfrac{1}{6}k(k+1)(2k+1).$$

We want to prove that the result is true for $n = k + 1$, i.e. that

$$\begin{aligned}1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 &= \tfrac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)\\ &= \tfrac{1}{6}(k+1)(k+2)(2k+3).\end{aligned}$$

Using the assumed result for $n = k$ gives

$$\begin{aligned}1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 &= \tfrac{1}{6}k(k+1)(2k+1) + (k+1)^2\\ &= \tfrac{1}{6}(k+1)[k(2k+1) + 6(k+1)]\\ &= \tfrac{1}{6}(k+1)(2k^2 + k + 6k + 6)\\ &= \tfrac{1}{6}(k+1)(2k^2 + 7k + 6)\\ &= \tfrac{1}{6}(k+1)(k+2)(2k+3).\end{aligned}$$

Add $(k+1)^2$ to each side.

Take out $= \tfrac{1}{6}(k+1)$ as a factor – this is part of the target expression.

Step 3 So if the result is true when $n = k$, then it is true when $n = k + 1$.
As it is true for $n = 1$, it is true for all $n \geqslant 1$ by induction.

Sums of Integers

Appendix 10. An Example of Proof by Induction

Mathematicians find it useful to have neat formulae which give the sum of various lists of numbers. In this case the challenge is to find a formula which gives the sum of the first n counting numbers.

For example, the sum of just the first number is 1, the sum of the first two numbers is 3 (i.e. $1 + 2$), the sum of the first three numbers is 6 (i.e. $1 + 2 + 3$), the sum of the first four numbers is 10 (i.e. $1 + 2 + 3 + 4$), and so on.

A candidate formula which seems to describe this pattern is:

$$\text{Sum}(n) = \tfrac{1}{2}n(n + 1).$$

In other words if we want to find the sum of the first n numbers, then we simply enter that number into the formula above and work out the answer.

Proof by induction can prove that this formula works for every number up to infinity.

The first step is to show that the formula works for the first case, $n = 1$. This is fairly straightforward, because we know that the sum of just the first number is 1, and if we enter $n = 1$ into the candidate formula we get the correct result:

$$\text{Sum}(n) = \tfrac{1}{2}n(n + 1)$$

$$\text{Sum}(1) = \tfrac{1}{2} \times 1 \times (1 + 1)$$

$$\text{Sum}(1) = \tfrac{1}{2} \times 1 \times 2$$

$$\text{Sum}(1) = 1.$$

The first domino has been toppled.

The next step in proof by induction is to show that if the formula is true for any value n, then it must also be true for $n + 1$. If

$$\text{Sum}(n) = \tfrac{1}{2}n(n + 1),$$

then,

$$\text{Sum}(n + 1) = \text{Sum}(n) + (n + 1)$$

$$\text{Sum}(n + 1) = \tfrac{1}{2}n(n + 1) + (n + 1).$$

After rearranging and regrouping the terms on the right, we get

$$\text{Sum}(n + 1) = \tfrac{1}{2}(n + 1)\ [(n + 1) + 1].$$

What is important to note here is that the form of this new equation is exactly the same as the original equation except that every appearance of n has been replaced by $(n + 1)$.

In other words, if the formula is true for n, then it must also be true for $n + 1$. If one domino falls, it will always knock over the next one. The proof by induction is complete.

Sums of Integers

Sometimes in mathematics, an infinite process can be involved in the actual structure of the logical reasoning itself. This happens, for example, with a powerful method called *proof by induction.*

The general idea of this is not unlike a railway train; a lot of coaches are coupled together, an engine pulls on the first one, this coach then pulls on the second, and so on until the whole train moves.

Here's an example. There is a simple formula for the sum of the first n whole numbers:

$$1 + 2 + 3 + 4 + \ldots + n = \tfrac{1}{2}n(n+1).$$

According to this formula, then, the sum of the first 10 whole numbers is $\frac{1}{2} \times 10 \times 11 = 55$, and it is easy to check by direct summation that this is correct. But how can we prove that the formula is correct for *any* whole number n?

Well, suppose for a moment that we knew it to be true for some *particular* whole number $n = p$. If that were so, we could then deduce, simply by adding one more term, that the sum of the first $p + 1$ whole numbers must be:

$$1 + 2 + 3 + 4 + \ldots + p + (p+1) = \tfrac{1}{2}p(p+1) + (p+1).$$

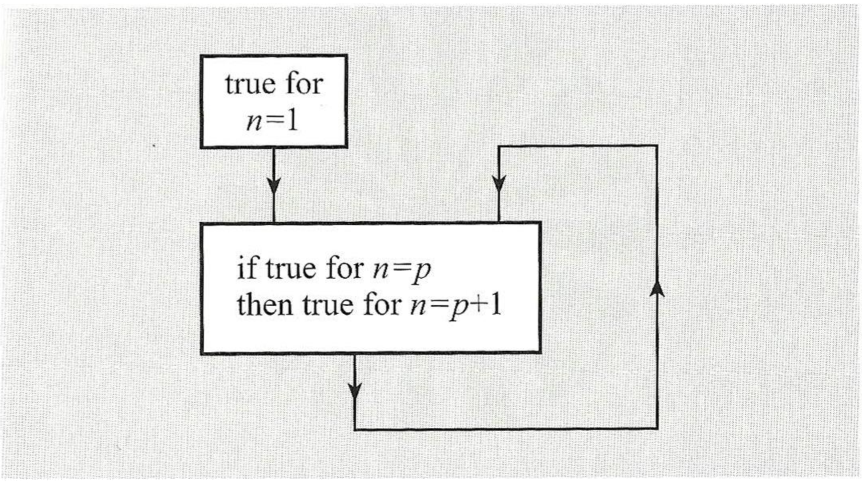
And there is something very interesting about the right-hand side of this equation: with a little algebra we can rewrite it in the form $\frac{1}{2}(p+1)\,(p+2)$.

But this is just the original formula $\frac{1}{2}n(n+1)$ *with* $n = p+1$ *instead of* $n = p$!

We have shown, in other words, that *if* the formula happened to be true for one particular whole number n, *then it would be true for the next one as well.*

At this stage we have, so to speak, coupled all the coaches, and the final step is to start the engine.

And to do this, we simply observe that the formula certainly works when $n = 1$, because the 'sum' then has only one term, 1, and $\frac{1}{2}n(n+1) = \frac{1}{2} \times 1 \times 2$, which is,



The idea of proof by induction.

indeed, 1. From what we have just shown, then, the formula must also be true for $n = 2$ as well, and, in the same way, because it is true when $n = 2$ it must also be true when $n = 3$, and so on.

So the sum of the first n whole numbers is $\frac{1}{2}n(n+1)$, for all positive whole numbers n.

And while there are other, equally attractive, ways of proving this particular result, the whole idea of proof by induction is a very general one, and finds application, from time to time, in countless different branches of mathematics, even at the highest level.

Euler's Formula for Polyhedra

Much ado about knotting

Euler's Formula for Polyhedra

number of faces — number of edges — number of vertices

$$F - E + V = 2$$

What does it say?

The numbers of faces, edges, and vertices of a solid are not independent, but are related in a simple manner.

Why is that important?

It distinguishes between solids with different topologies using the earliest example of a topological invariant. This paved the way to more general and more powerful techniques, creating a new branch of mathematics.

What did it lead to?

One of the most important and powerful areas of pure mathematics: topology, which studies geometric properties that are unchanged by continuous deformations. Examples include surfaces, knots, and links. Most applications are indirect, but its influence behind the scenes is vital. It helps us understand how enzymes act on DNA in a cell, and why the motion of celestial bodies can be chaotic.

Topology is often characterised as 'rubber-sheet geometry' because it is the kind of geometry that would be appropriate for figures drawn on a sheet of elastic, so that lines can bend, shrink, or stretch, and circles can be squashed so that they turn into triangles or squares. All that matters is *continuity*: you are not allowed to rip the sheet apart. It may seem remarkable that anything so weird could have any importance, but continuity is a basic aspect of the natural world and a fundamental feature of mathematics. Today we mostly use topology indirectly, as one mathematical technique among many. You don't find anything obviously topological in your kitchen. However, a Japanese company did market a chaotic dishwasher, which according to their marketing people cleaned dishes more efficiently, and our understanding of chaos rests on topology. So do some important aspects of quantum field theory and that iconic molecule DNA. But, when Descartes counted the most obvious features of the regular solids and noticed that they were not independent, all this was far in the future.

It was left to the indefatigable Euler, the most prolific mathematician in history, to prove and publish this relationship, which he did in 1750 and 1751. I'll sketch a modern version. The expression $F-E+V$ may seem fairly arbitrary, but it has a very interesting structure. Faces (F) are polygons, of dimension 2, edges (E) are lines, so have dimension 1, and vertices (V) are points, of dimension 0. The signs in the expression alternate, $+-+$, with $+$being assigned to features of even dimension and $-$ to those of odd dimension. This implies that you can simplify a solid by merging its faces or removing edges and vertices, and these changes will not alter the number $F-E+V$ provided that every time you get rid of a face you also remove an edge, or every time you get rid of a vertex you also remove an edge. The alternating signs mean that changes of this kind cancel out.

Now I'll explain how this clever structure makes the proof work. Figure 21 shows the key stages. Take your solid. Deform it into a nice round sphere, with its edges being curves on that sphere. If two faces meet along a common edge, then you can remove that edge and merge the faces into one. Since this merger reduces both F and E by 1, it doesn't change $F-E+V$. Keep doing this until you get down to a single face, which covers almost all of the sphere. Aside from this face, you are left with only edges and vertices. These must form a tree, a network with no closed loops, because any closed loop on a sphere separates at least two faces: one inside it, the other outside it. The branches of this tree are the remaining edges of the solid, and they join together at the remaining vertices. At this stage only one face remains: the entire sphere, minus the tree. Some branches of this tree connect to other branches at both ends, but some, at the extremes, terminate in a vertex, to which no other branches attach. If you remove one of these terminating branches together with that vertex, then the tree gets smaller, but since both E and V decrease by 1, $F-E+V$ again remains unchanged.

This process continues until you are left with a single vertex sitting on an otherwise featureless sphere. Now $V=1$, $E=0$, and $F=1$. So $F-E+V=1-0+1=2$. But since each step leaves $F-E+V$ unchanged, its value at the beginning must also have been 2, which is what we want to prove.

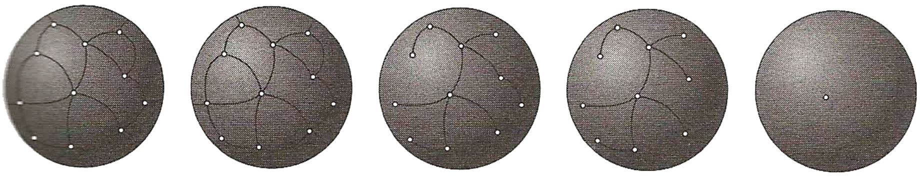


Fig 21 Key stages in simplifying a solid. *Left to right*: (1) Start. (2) Merging adjacent faces. (3) Tree that remains when all faces have been merged. (4) Removing an edge and a vertex from the tree. (5) End.

It's a cunning idea, and it contains the germ of a far-reaching principle. The proof has two ingredients. One is a simplification process: remove either a face and an adjacent edge or a vertex and an edge that meets it. The other is an invariant, a mathematical expression that remains unchanged whenever you carry out a step in the simplification process. Whenever these two ingredients coexist, you can compute the value of the invariant for any initial object by simplifying it as far as you can, and then computing the value of the invariant for this simplified version. Because it is an invariant, the two values must be equal. Because the end result is simple, the invariant is easy to calculate.

Now I have to admit that I've been keeping one technical issue up my sleeve. Descartes's formula does not, in fact, apply to any solid. The most familiar solid for which it fails is a picture frame. Think of a picture frame made from four lengths of wood, each rectangular in cross-section, joined at the four corners by 45° mitres as in Figure 22 (*left*). Each length of wood contributes 4 faces, so $F=16$. Each length also contributes 4 edges, but the mitre joint creates 4 more at each corner, so $E=32$. Each corner comprises 4 vertices, so $V=16$. Therefore $F-E+V=0$.

What went wrong?

Induction Questions 1

1 A sequence is defined by $u_{n+1} = 3u_n + 2$, $u_1 = 2$.
Prove by induction that $u_n = 3^n - 1$.

2 A sequence is defined by $u_{n+1} = 2u_n - 1$, $u_1 = 2$.
Prove by induction that $u_n = 2^{n-1} + 1$.

3 A sequence is defined by $u_{n+1} = 4u_n - 6$, $u_1 = 3$.
Prove by induction that $u_n = 4^{n-1} + 2$.

4 A sequence is defined by $u_{n+1} = \dfrac{u_n}{u_n + 1}$, $u_1 = 1$.

(i) Find the values of u_2, u_3 and u_4.

(ii) Suggest a general formula for u_n, and prove your conjecture by induction.

5 A sequence of integers $u_1, u_2, u_3, \ldots$ is defined by

$$u_1 = 5 \text{ and } u_{n+1} = 3u_n - 2^n \text{ for } n \geqslant 1.$$

(i) Use this definition to find u_2 and u_3.

(ii) Prove by induction that $u_n = 2^n + 3^n$ for all positive integers n.

[MEI, *part*]

6 A sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_1 = \tfrac{7}{2} \text{ and } u_n = \tfrac{1}{2}u_{n-1} + n^2 \text{ for } n \geqslant 2.$$

Prove by induction that $u_n = 2n^2 - 4n + 6 - \left(\tfrac{1}{2}\right)^n$ and for all positive integers n.

[MEI, *part*]

7 Prove, using the method of mathematical induction, that $2^{4n+1} + 3$ is a multiple of 5 for any positive integer n.

[MEI, *part*]

8 Prove that $11^{n+2} + 12^{2n+1}$ is divisible by 133 for $n \geqslant 0$.

9 You are given the matrix $\mathbf{A} = \begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix}$.

(i) Calculate $\mathbf{A}^2$ and $\mathbf{A}^3$.

(ii) Show that the formula $\mathbf{A}^n = \begin{pmatrix} 1 - 2n & -4n \\ n & 1 + 2n \end{pmatrix}$ is consistent with the given value of $\mathbf{A}$ and your calculations for $n = 2$ and $n = 3$.

(iii) Prove by induction that the formula for $\mathbf{A}^n$ is correct when n is a positive integer.

[MEI, *part*]

Induction Questions 2

In questions 1 *to* 12, *prove the result given by induction.*

1 $1 + 3 + 5 + \ldots + (2n - 1) = n^2$

(This was the first example of proof by induction ever published, by Francesco Maurolycus in 1575.)

2 $1 + 2 + 3 + \ldots + n = \frac{1}{2}n(n + 1)$

3 $2 + 2^2 + 2^3 + \ldots + 2^n = 2(2^n - 1)$

4 $\displaystyle\sum_{k=1}^{n} k^3 = \frac{1}{4}n^2(n + 1)^2$

5 $(1 \times 2) + (2 \times 3) + (3 \times 4) + \ldots + n(n + 1) = \frac{1}{3}n(n + 1)(n + 2)$

6 $\displaystyle\sum_{k=0}^{n-1} x^k = \frac{1 - x^n}{1 - x} \qquad (x \neq 1)$

7 $(1 \times 2 \times 3) + (2 \times 3 \times 4) + \ldots + n(n + 1)(n + 2) = \frac{1}{4}n(n + 1)(n + 2)(n + 3)$

8 $\displaystyle\sum_{k=1}^{n} (3k + 1) = \frac{1}{2}n(3n + 5)$

9 $\dfrac{1}{3} + \dfrac{1}{15} + \dfrac{1}{35} + \ldots + \dfrac{1}{4n^2 - 1} = \dfrac{n}{2n + 1}$

10 $\dfrac{1}{1 \times 2 \times 3} + \dfrac{1}{2 \times 3 \times 4} + \ldots + \dfrac{1}{n(n + 1)(n + 2)} = \dfrac{n(n + 3)}{4(n + 1)(n + 2)}$

11 $\left(1 - \dfrac{1}{2^2}\right)\left(1 - \dfrac{1}{3^2}\right)\left(1 - \dfrac{1}{4^2}\right)\ldots\left(1 - \dfrac{1}{n^2}\right) = \dfrac{n + 1}{2n} \qquad$ for $n \geqslant 2$

12 $1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + n \times n! = (n + 1)! - 1$

(Remember: $n!$ means $n(n - 1)(n - 2)\ldots 3 \times 2 \times 1$.)