$\sqrt{2}$ is Irrational

1. Assume $\sqrt{2}$ is rational, i.e. $\sqrt{2} = \frac{a}{b}$

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2. Where $\frac{a}{b}$ is a fraction in lowest terms, i.e. cancelled down as far as possible - IMPORTANT

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3. Square both sides…

$$2 = \frac{a^2}{b^2}$$

4. Multiply up…

$$2b^2 = a^2$$

5. a^2 must be even since $= 2b^2$

6. And therefore a^2 is even

7. And therefore a is even (because even x even = even)

This itself needs proving before we should continue

8. And therefore a can be written as $2\times$ another number…

$$a = 2p$$

9. And therefore…

$$a^2 = (2p)^2 = 4p^2$$

10. And therefore…

$$2b^2 = 4p^2$$

11. And therefore…

$$b^2 = 2p^2$$

12. And therefore b is also an even number.

13. And therefore a and b are both even and are both divisible by two, i.e. they have a common factor of 2.

14. But we've already stated that…

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Where $\frac{a}{b}$ is a fraction in lowest terms, i.e. cancelled down as far as possible - IMPORTANT

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15. So we have a contradiction.

16. And therefore, our original assumption must be wrong…

$$\sqrt{2} \text{ is NOT rational but is irrational.}$$

17. We've proved by contradiction that $\sqrt{2}$ is NOT rational.

$\sqrt{5}$ is Irrational

1. Assume $\sqrt{5}$ is rational, i.e. $\sqrt{5} = \frac{a}{b}$

   2. Where $\frac{a}{b}$ is a fraction in lowest terms, i.e. cancelled down as far as possible - IMPORTANT

3. Square both sides…

$$5 = \frac{a^2}{b^2}$$

4. Multiply up…

$$5b^2 = a^2$$

5. a^2 must be multiple of 5 since $= 5b^2$

6. And therefore a^2 is a multiple of 5

7. And therefore a is a multiple of 5 (because if a^2 is multiple of 5 then a is multiple of 5)

   This itself needs proving before we should continue

8. And therefore a can be written as 5× another number…

$$a = 5p$$

9. And therefore…

$$a^2 = (5p)^2 = 25p^2$$

10. And therefore…

$$5b^2 = 25p^2$$

11. And therefore…

$$b^2 = 5p^2$$

12. And therefore b is also an even number.

13. And therefore a and b are both multiples of 5 and are both divisible by five, i.e. they have a common factor of 5.

14. But we've already stated that…

   Where $\frac{a}{b}$ is a fraction in lowest terms, i.e. cancelled down as far as possible - IMPORTANT

15. So we have a contradiction.

16. And therefore, our original assumption must be wrong…

   $\sqrt{5}$ is NOT rational but is irrational.

17. We've proved by contradiction that $\sqrt{5}$ is NOT rational.

Why doesn't this argument work for $\sqrt{4}, \sqrt{9}$ etc.?

Here's why this argument doesn't work for $\sqrt{4}, \sqrt{9}$ etc.

For roots of non-square numbers…

1. If a^2 is a square number and multiple of 2 (even) then a is multiple of 2 (even)

$$even \times even = even$$

Both factors must be same, and therefore…

$$a^2 = (2 \times \dots) \times (2 \times \dots)$$

2. If a^2 is a square number and multiple of 5 then a is multiple of 5

$$a^2 = \_ \times \_$$

Both factors must be same, and therefore…

$$a^2 = (5 \times \dots) \times (5 \times \dots)$$

However, for square numbers…

3. If a^2 is a square number and multiple of 4 then a is multiple of 4.

Counter example:

$$a^2 = 100 \qquad \text{(is a multiple of 4)}$$
$$\Rightarrow a = 10 \qquad \text{(not a multiple of 4)}$$
$$a^2 = (2 \times 5) \times (2 \times 5)$$

4. If a^2 is a square number and multiple of 9 then a is multiple of 9.

Counter example:

$$a^2 = 36 \qquad \text{(is a multiple of 4)}$$
$$\Rightarrow a = 6 \qquad \text{(not a multiple of 4)}$$
$$a^2 = (3 \times 2) \times (3 \times 2)$$

With square numbers, the square root is distributed across both factors instead of appearing in its entirety, and repeated, in both factors.

Therefore, the proof breaks down. There is no contradiction, the assumption remains correct and the conjecture that was to be proven remains unproven.

Intro Task

1) I'm thinking of a square number, a^2, which is also a multiple of 2. What can you say about the number a?

2) I'm thinking of a square number, a^2, which is also a multiple of 5. What can you say about the number a?

3) I'm thinking of a square number, a^2, which is also a multiple of any prime number. What can you say about the number a?

4) I'm thinking of a square number, a^2, which is also a multiple of 4. What can you say about the number a?
   Can you give an example (counter-example?) to support your ideas?

5) I'm thinking of a square number, a^2, which is also a multiple of 9. What can you say about the number a?
   Can you give an example (counter-example?) to support your ideas?

6) I'm thinking of a square number, a^2, which is also a multiple of 8. What can you say about the number a?

7) I'm thinking of a square number, a^2, which is also a multiple of 12. What can you say about the number a?

8) I'm thinking of a square number, a^2, which is also a multiple of any non-square, non-prime number. What can you say about the number a?