Standard Integrals

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = sin^{-1}\left(\frac{x}{a}\right) + c$$

Observe that...

$cos\theta = \sqrt{1 - sin^2\theta}$	is useful for	$\sqrt{1 - x^2}$
$sec^2\theta = 1 + tan^2\theta$	is useful for	$1 + x^2$
$tan^2\theta = sec^2\theta - 1$	is useful for	$x^2 - 1$

Proof of $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

Step 1: Proof of $\int \frac{1}{1+x^2}\,dx = \tan^{-1}x$

$$y = \tan^{-1}x$$

$$\Rightarrow x = \tan y$$

Using the fact that $y = \tan x \Longrightarrow \frac{dy}{dx} = \sec^2 x$, the product rule gives:

$$\frac{dx}{dy} = \sec^2 y = 1 + \tan^2 y = 1 + x^2$$

(because $\sec^2\theta = 1 + \tan^2\theta$ and $\tan y = x$)

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2}$$

We have the result that when:

$$y = \tan^{-1}x$$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

And…

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}x$$

Step 2: Proof of $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

This follows as an extension by integration by substitution:

$$\int \frac{1}{a^2 + x^2}\,dx$$

$$= \int \frac{1}{a^2\left[1 + \left(\frac{x}{a}\right)^2\right]}\,dx$$

$$= \frac{1}{a}\int \frac{1}{a\left[1 + \left(\frac{x}{a}\right)^2\right]}\,dx$$

$$= \frac{1}{a}\int \frac{1}{a} \times \frac{1}{\left[1 + \left(\frac{x}{a}\right)^2\right]}\,dx$$

Using the substitution $u = \frac{x}{a}$, where $\frac{du}{dx} = \frac{1}{a}$:

$$= \frac{1}{a}\int \frac{1}{[1 + u^2]}\frac{du}{dx}\,dx$$

$$= \frac{1}{a}\int \frac{1}{[1 + u^2]}\,du$$

$$= \frac{1}{a}tan^{-1}u$$

$$= \frac{1}{a}tan^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}tan^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}tan^{-1}\left(\frac{x}{a}\right)$$

Proof of $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = sin^{-1}\left(\frac{x}{a}\right) + c$

Step 1: Proof of $\int \frac{1}{\sqrt{1-x^2}}\,dx = sin^{-1}x$

$$y = sin^{-1}x$$
$$\Rightarrow x = siny$$

$y = sinx$, $\frac{dy}{dx} = cosx$ gives:

$$\frac{dx}{dy} = cosy = \sqrt{1 - sin^2 y} = \sqrt{1 - x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

We have the result that when:

$$y = sin^{-1}x$$
$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

And…

$$\int \frac{1}{\sqrt{1 - x^2}}\,dx = sin^{-1}x$$

Step 2: Proof of $\int \frac{1}{\sqrt{a^2-x^2}}dx = sin^{-1}\left(\frac{x}{a}\right)$

This follows as an extension by integration by substitution:

$$\int \frac{1}{\sqrt{a^2-x^2}}dx$$

$$= \int \frac{1}{\sqrt{a^2\left[1-\left(\frac{x}{a}\right)^2\right]}}dx = \int \frac{1}{a\sqrt{\left[1-\left(\frac{x}{a}\right)^2\right]}}dx$$

$$= \int \frac{1}{a\sqrt{\left[1-\left(\frac{x}{a}\right)^2\right]}}dx$$

$$= \int \frac{1}{a} \times \frac{1}{\sqrt{\left[1-\left(\frac{x}{a}\right)^2\right]}}dx$$

Using the substitution $u = \frac{x}{a}$, where $\frac{du}{dx} = \frac{1}{a}$:

$$= \int \frac{1}{\sqrt{[1-u^2]}}\frac{du}{dx}dx$$

$$= \int \frac{1}{\sqrt{[1-u^2]}}du$$

$$= sin^{-1}u$$

$$= sin^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = sin^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = sin^{-1}\left(\frac{x}{a}\right)$$

Some Points to Note

Angles in radians

Standard integrals cannot be used as in example below;

$$\int \frac{1}{1+25x^2}\,dx = \int \frac{1}{1+(5x)^2}\,dx$$

Instead do;

$$\int \frac{1}{1+25x^2}\,dx = \int \frac{1}{25\left(\frac{1}{25}+x^2\right)}\,dx$$
$$= \frac{1}{25}\int \frac{1}{\left(\frac{1}{25}+x^2\right)}\,dx$$

A second example;

$$\int \frac{72}{\sqrt{1-81x^2}}\,dx$$
$$= \int \frac{72}{\sqrt{81\left(\frac{1}{81}+x^2\right)}}\,dx$$
$$= \int \frac{72}{\sqrt{81}\sqrt{\left(\frac{1}{81}+x^2\right)}}\,dx$$
$$= \int \frac{72}{9\sqrt{\left(\frac{1}{81}+x^2\right)}}\,dx$$
$$= \frac{72}{9}\int \frac{1}{\sqrt{\left(\frac{1}{81}+x^2\right)}}\,dx$$
$$= 8\int \frac{1}{\sqrt{\left(\frac{1}{81}+x^2\right)}}\,dx$$