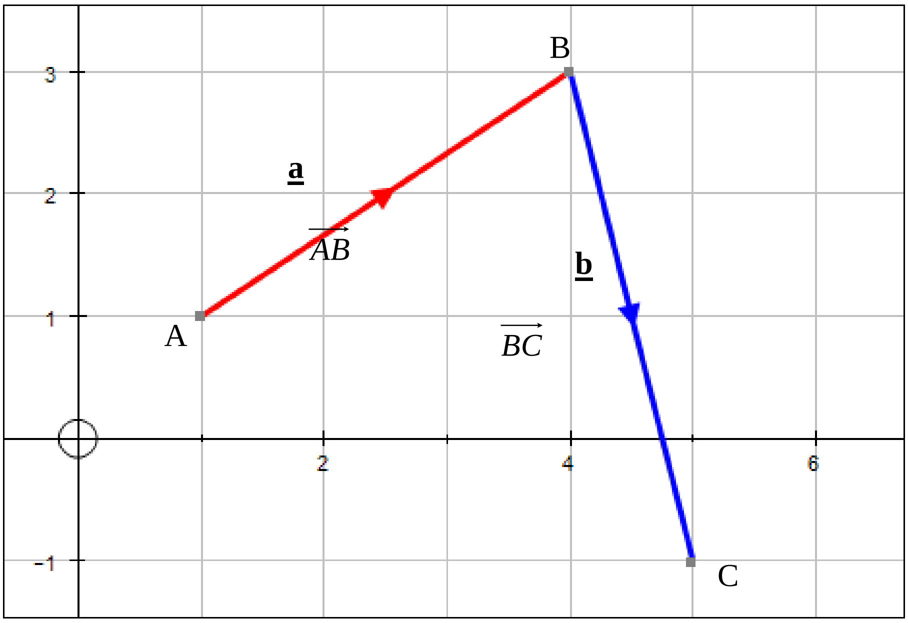
Vectors

Think of a vector like someone giving you directions ("go east for 2 miles") or like a path.

They have a **direction** and a **size**.



In the illustration above, the red vector (the one on the left) goes from A to B. We can call it **a** (or $\overrightarrow{AB}$) and the blue vector (the one on the right) we'll call **b** (or $\overrightarrow{BC}$).

Vectors are usually written as '**column vectors**'. So:

$$\underline{\mathbf{a}} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \qquad \underline{\mathbf{b}} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \qquad \text{In general } \begin{pmatrix} x \\ y \end{pmatrix}$$

They can also be written in i, j, k notation. So:

$$\underline{\mathbf{a}} = 3i+2j \qquad \underline{\mathbf{b}} = i - 4j \qquad \text{In general } i + j$$

The vector **a** takes us on a path from the coordinates (1,1) to (4,3).

To go the **opposite way**, i.e. from (4, 3) to (1, 1) would be **−a**, (or $-\overrightarrow{AB}$) or $\begin{pmatrix} -3 \\ -2 \end{pmatrix}$.

We can **add vectors**. Imagine following the vectors **a** and then **b**. We'd begin at (1,1) and end at (5,-1). This is a vector of:

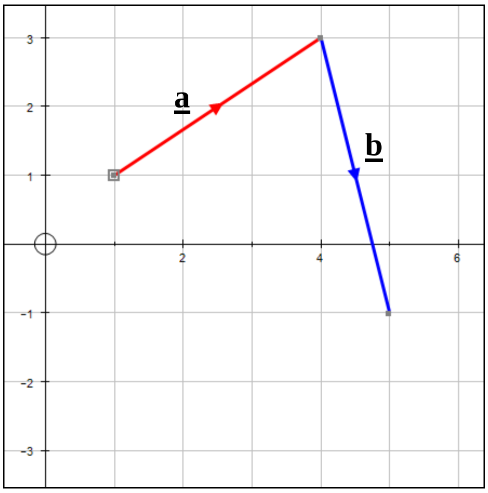
$$\begin{pmatrix} 5-1 \\ -1-1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

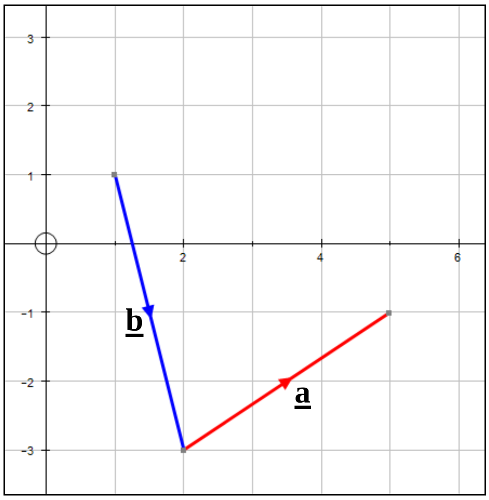
We get the same result by adding the vectors:

$$\underline{\mathbf{a}}+\underline{\mathbf{b}} \text{ (or } \overrightarrow{AB} + \overrightarrow{BC}\text{)} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ -4 \end{pmatrix} = \overrightarrow{AC} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

The answer, $\overrightarrow{AC}$, $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$, we call the 'resultant vector'. It's the result of doing $\underline{\mathbf{a}}+\underline{\mathbf{b}}$ (or $\overrightarrow{AB}$ then $\overrightarrow{BC}$).

Furthermore, just like adding, it's probably obvious that $\underline{\mathbf{a}}+\underline{\mathbf{b}} = \underline{\mathbf{b}}+\underline{\mathbf{a}}$. They both end up at the same point even though they took different routes there:





This is not so with subtracting, but you already know that. (e.g. $5 - 2 \neq 2 - 5$)

We can **multiply a vector by a scalar**:

$$2\underline{\mathbf{a}} \text{ (or } 2\overrightarrow{AB}\text{)} = 2\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

Notice same direction, twice the size (hmm, size, see further down).

Obviously we can do combinations of these, such as:

$$3\underline{\mathbf{a}} - 2\underline{\mathbf{b}} \text{ (or } 3\overrightarrow{AB} - 2\overrightarrow{BC}\text{)} = 3\begin{pmatrix} 3 \\ 2 \end{pmatrix} - 2\begin{pmatrix} 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \end{pmatrix} - \begin{pmatrix} 2 \\ -8 \end{pmatrix} = \begin{pmatrix} 7 \\ 14 \end{pmatrix}$$

Continues…

Parallel Vectors

If two vectors are parallel then one is a multiple of the other. The following vectors are all parallel…

$$\begin{pmatrix}3\\2\end{pmatrix} \qquad \begin{pmatrix}6\\4\end{pmatrix} \qquad \begin{pmatrix}15\\10\end{pmatrix} \qquad \begin{pmatrix}-3\\-2\end{pmatrix} \qquad \begin{pmatrix}1.5\\1\end{pmatrix}$$

Magnitude of a Vector (i.e. Size of a Vector) For 'magnitude', read 'size'.

If a vector was a path it may be nice to know how long the path is. Same with vectors, and using Pythagoras we can work this out. The size of **a**, called the 'modulus' of **a**, denoted by $|\mathbf{a}|$, is found by:

$$|\mathbf{a}| = |\overrightarrow{AB}| = \sqrt{3^2+2^2} = \sqrt{13}$$

Be careful, the size of **a**+**b** will be the size of $|(\mathbf{a}+\mathbf{b})|$, not $|\mathbf{a}|+|\mathbf{b}|$. I.e.:

$$|(\mathbf{a}+\mathbf{b})| = \left|\begin{pmatrix}3\\2\end{pmatrix}+\begin{pmatrix}1\\-4\end{pmatrix}\right| = \left|\begin{pmatrix}4\\-2\end{pmatrix}\right| = \sqrt{4^2+-2^2} = \sqrt{20} = \sqrt{4}\sqrt{5} = 2\sqrt{5}$$

*Using knowledge of surds.

The **midpoint of a vector** is halfway along it, i.e. half of it. E.g. the midpoint of **a** is:

$$\frac{1}{2}\mathbf{a} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}1.5\\1\end{pmatrix}$$

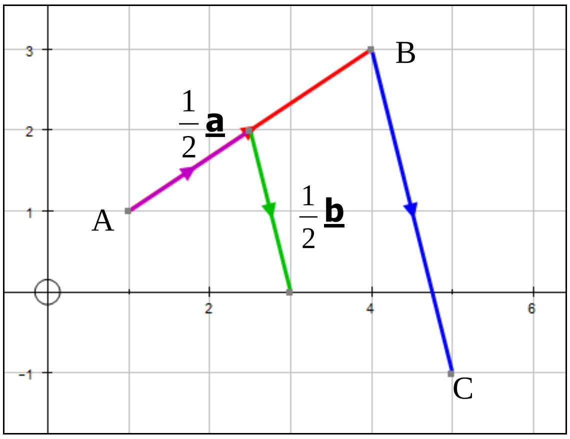
Using this idea we can find the midpoint of **a**+**b**, or $\overrightarrow{AC}$ (see how this is handy now?):

$$\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b} = \frac{1}{2}\begin{pmatrix}3\\2\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\-4\end{pmatrix} = \begin{pmatrix}1.5\\1\end{pmatrix} + \begin{pmatrix}0.5\\-2\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix}$$

Or using $\overrightarrow{AC}$:

$$\frac{1}{2}\begin{pmatrix}4\\-2\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix}$$

In pictures:

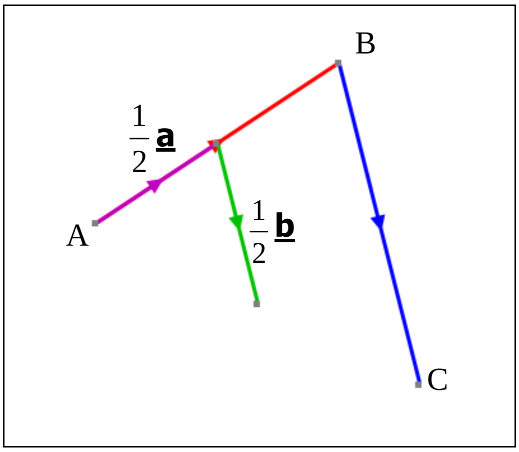


The purple vector is $\frac{1}{2}$ **a**, the green vector is $\frac{1}{2}$ **b**. The midpoint is $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ away from A (and $\begin{pmatrix} -2 \\ +1 \end{pmatrix}$ away from C, i.e. it's in the middle!). Be careful though. You'll see that *(2 -1)* does not represent the coordinates of the midpoint of $\overrightarrow{AC}$. To find the coordinates of the midpoint of $\overrightarrow{AC}$ we need to add $\frac{1}{2}$ **a** + $\frac{1}{2}$ **b** onto OA:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

Reducing to vectors in terms of a and b.

If we consider vectors as paths, labelled as **a**, **b** etc, and we do calculations in terms of **a** and **b** then we don't actually need the coordinate grid.



The illustration above shows the midpoint of $\overrightarrow{AC}$ as in the previous example but without the grid. We can show that this is the midpoint by the calculation:

$$\frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}\overrightarrow{AC}$$

A new dimension. We can extend our learning of vectors to include 3 dimensional vectors, such as:

$$\underline{\mathbf{p}} = \begin{pmatrix} 4 \\ -4 \\ 3 \end{pmatrix} \qquad \underline{\mathbf{q}} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \qquad \text{In general} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ or } i + j + k$$

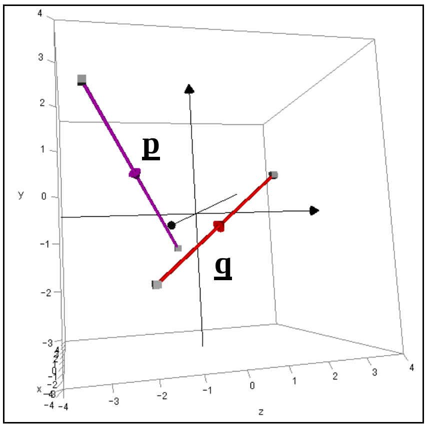
We can add, subtract, multiply, find the midpoint etc of these vectors just as we did with the 2D ones.

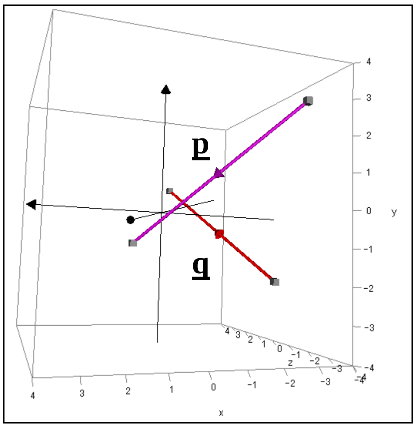
$$\underline{\mathbf{p}}+\underline{\mathbf{q}} = \begin{pmatrix} 4 \\ -4 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ 7 \end{pmatrix}$$

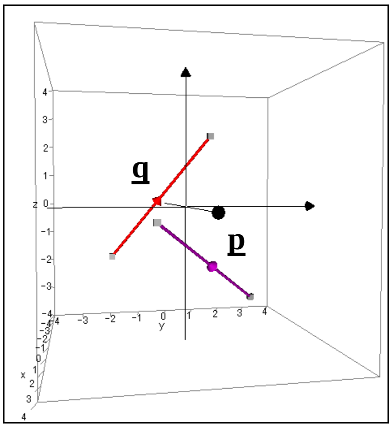
$$3\underline{\mathbf{p}} - 2\underline{\mathbf{q}} = 3\begin{pmatrix} 4 \\ -4 \\ 3 \end{pmatrix} - 2\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 12 \\ -12 \\ 9 \end{pmatrix} - \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} = \begin{pmatrix} 8 \\ -18 \\ 17 \end{pmatrix}$$

$$\text{Midpoint of } \underline{\mathbf{p}}+\underline{\mathbf{q}} = \frac{1}{2}(\underline{\mathbf{p}}+\underline{\mathbf{q}}) = \frac{1}{2}\begin{pmatrix} 6 \\ -1 \\ 7 \end{pmatrix} = \begin{pmatrix} 3 \\ -0.5 \\ 3.5 \end{pmatrix}$$

3D vectors are helpful because they still operate in the same way as 2D vectors whilst illustrations of them become difficult to understand. The 3 images below show the same vectors, **p** and **q**, from different points of view:







2D representations of 3D concepts do not work well!

Graphing software, such as Autograph (used here), illustrate 3D vectors well as the POV can be moved, rotated and zoomed to provide a better idea of what's happening. They can also be fun to play with, although not so much as a Nintendo Wii! (Hmm, computer games, very much based on vectors themselves).

That's it, for GCSE, hurrah!

Or for A level see the next page…

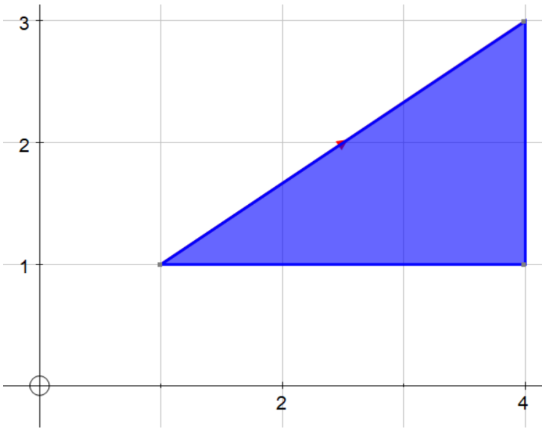
The Unit Vector

The unit vector is the vector in a particular direction and of just **one-unit in length**. To find this, we take our original vector values and divide by the size of the vector.

$$3i + 2j \Rightarrow \frac{3}{\sqrt{13}}i + \frac{2}{\sqrt{13}}j$$

Angle between a vector and the positive x-axis

To find the angle between a vector and the x-axis, we consider the vector as a triangle



Where…

$$tan\theta = \frac{opp}{adj} = tan^{-1}\left(\frac{2}{3}\right) = 33.6°$$

Magnitude-Direction Form (aka Polar Form)

Angles measured anti-clockwise from +ve x-axis.

Component Form	Magnitude-Direction Form
$\binom{3}{2} = 3i + 2j$	$\left(\sqrt{3^2 + 2^2}, tan^{-1}\left(\frac{2}{3}\right)\right) = (\sqrt{13}, 33.69°)$

For notes on extension material see
http://www.colmanweb.co.uk/Assets/Resources/VectorScalarProducts.doc