

What's this $e^{i\pi}$ thing?

Using Maclaurin Expansions of e^x , $\sin\theta$ and $\cos\theta$ together with $i^2 = -1$, we derive the following...

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \dots \\ &= 1 + i\theta + \frac{-\theta^2}{2!} + \frac{-i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \dots \end{aligned}$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$$

$$i\sin\theta = i\theta - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} - \frac{i\theta^7}{7!} + \dots$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots$$

$$\begin{aligned} \cos\theta + i\sin\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots + i\theta - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} - \frac{i\theta^7}{7!} + \dots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \dots \\ &= e^{i\theta} \end{aligned}$$

$$r(\cos\theta + i\sin\theta) = re^{i\theta}$$

For the case where $r = 1$ and $\theta = \pi$...

$$(\cos\pi + i\sin\pi) = (-1 + 0) = e^{i\pi} = -1$$

And hence...

$$e^{i\pi} = -1$$