FP3 Second Order Differential Equation Answers

1(a)	$(m+1)^2 = -1$	M1		Completing sq or formula
	$m = -1 \pm i$	A1	2	
(b)(i)	CF is $e^{-x}(A\cos x + B\sin x)$	M1		If m is real give M0
	{or $e^{-x}A\cos(x+B)$ **but not** $Ae^{(-1+i)x} + Be^{(-1-i)x}$}	A1√		On wrong a's and b's but roots must be complex.
	{P.Int.} try $y = px + q$	M1		OE
	$2p + 2(px+q) = 4x$	A1		
	$p = 2,\ \ q = -2$	A1√		On one slip
	GS $y = e^{-x}(A\cos x + B\sin x) + 2x - 2$	B1√	6	Their CF + their PI with two arbitrary constants.
(ii)	$x=0,\ y=1 \Rightarrow A = 3$	B1√		Provided an M1gained in (b)(i)
	$y'(x) = -e^{-x}(A\cos x + B\sin x) + e^{-x}(-A\sin x + B\cos x) + 2$	M1 A1√		Product rule used
	$y'(0) = 2 \Rightarrow 2 = -A + B + 2 \Rightarrow B = 3$	A1√		Slips
	$y = 3e^{-x}(\cos x + \sin x) + 2x - 2$		4	
	Total		**12**	

1(a)	$y = 2x + \sin 2x \Rightarrow y' = 2 + 2\cos 2x$ $\Rightarrow y'' = -4\sin 2x$	M1 A1		Need to attempt both y' and y''
	$-4\sin 2x - 5(2 + 2\cos 2x) + 4(2x + \sin 2x) = 8x - 10 - 10\cos 2x$	A1	3	CSO AG Substitute. and confirm correct
(b)	Auxiliary equation $m^2 - 5m + 4 = 0$	M1		
	$m = 4$ and 1	A1		
	CF: $Ae^{4x} + Be^{x}$	M1		
	GS: $y = Ae^{4x} + Be^{x} + 2x + \sin 2x$	B1√	4	Their CF + $2x$ + $\sin 2x$
(c)	$x = 0, y = 2 \Rightarrow \quad 2 = A + B$	B1√		Only ft if exponentials in GS
	$x = 0, y' = 0 \Rightarrow \quad 0 = 4A + B + 4$	B1√		Only ft if exponentials in GS and differentiated four terms at least
	Solving the simultaneous equations	M1		
	gives $A = -2$ and $B = 4$	A1	4	
	$y = -2e^{4x} + 4e^{x} + 2x + \sin 2x$			
	Total		**11**	

6(a)	$u=\frac{\mathrm{d}y}{\mathrm{d}x}+2y \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x}=\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\frac{\mathrm{d}y}{\mathrm{d}x}$	M1 A1		2 terms correct
	LHS of DE$\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x}-2\frac{\mathrm{d}y}{\mathrm{d}x}+4\frac{\mathrm{d}y}{\mathrm{d}x}+4y$			
	LHS: $\frac{\mathrm{d}u}{\mathrm{d}x}+2(u-2y)+4y$	M1		Substitution into LHS of DE as far as no derivatives of y
	$\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x}+2u=\mathrm{e}^{-2x}$	A1	4	CSO AG
(b)	IF is $\mathrm{e}^{\int 2\mathrm{d}x}=\mathrm{e}^{2x}$	B1		
	$\frac{\mathrm{d}}{\mathrm{d}x}\left[u\mathrm{e}^{2x}\right]=1$	M1 A1		
	$\Rightarrow u\mathrm{e}^{2x}=x+A$	A1		
	$\Rightarrow u=x\mathrm{e}^{-2x}+A\mathrm{e}^{-2x}$	A1	5	
	Alternative : Those using CF+PI Auxiliary equation,			
	$m+2=0\Rightarrow u_{CF}=A\mathrm{e}^{-2x}$	B1		
	For u_{PI} try $u_{PI}=kx\mathrm{e}^{-2x} \Rightarrow$	M1		
	$k\mathrm{e}^{-2x}-2kx\mathrm{e}^{-2x}+2kx\mathrm{e}^{-2x}\{=\mathrm{e}^{-2x}\}$	A1		LHS
	$\Rightarrow k=1\Rightarrow u_{PI}=x\mathrm{e}^{-2x}$	A1		
	$\Rightarrow u_{GS}=A\mathrm{e}^{-2x}+x\mathrm{e}^{-2x}$	A1		
(c)	$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}+2y=x\mathrm{e}^{-2x}+A\mathrm{e}^{-2x}$	M1		Use (b) to reach a 1st order DE in y and x
	IF is $\mathrm{e}^{\int 2\mathrm{d}x}=\mathrm{e}^{2x}$	B1		
	$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left[y\mathrm{e}^{2x}\right]=x+A$	A1√		
	$\Rightarrow y\mathrm{e}^{2x}=\frac{x^2}{2}+Ax+B$	A1√		
	$\Rightarrow y=\mathrm{e}^{-2x}\left(\frac{x^2}{2}+Ax+B\right)$	A1	5	
	Total		**14**	

5	Auxl. eqn $m^2-4m+3=0$	M1		PI
	$m=3$ and 1	A1		PI
	CF is $A\mathrm{e}^{3x}+B\mathrm{e}^{x}$	A1F		
	PI Try $y=a+b\sin x+c\cos x$	M1		Condone 'a' missing here
	$y'(x)=b\cos x-c\sin x$	A1		
	$y''(x)=-b\sin x-c\cos x$	A1F		ft can be consistent sign error(s)
	Substitute into DE gives	M1		
	$a=2$	B1		
	$4c+2b=5$ and $2c-4b=0$	A1		
	$b=0.5$,	A1F		ft a slip
	$c=1$	A1F		ft a slip
	GS: $y=A\mathrm{e}^{3x}+B\mathrm{e}^{x}+2+0.5\sin x+\cos x$	B1F	12	$y=$ candidate's CF and candidate's PI (must have exactly two arbitrary constants)
	Total		**12**	

1(a)	$y_{\text{PI}} = kx^2e^{5x} \Rightarrow y' = 2kxe^{5x} + 5kx^2e^{5x}$	M1 A1		Product rule to differentiate x^2e^{5x}
	$\Rightarrow y'' = 2ke^{5x} + 10kxe^{5x} + 10kxe^{5x} + 25kx^2e^{5x}$	A1ft		
	$\Rightarrow 2ke^{5x} + 20kxe^{5x} + 25kx^2e^{5x}$ $-10\left(2kxe^{5x} + 5kx^2e^{5x}\right) + 25kx^2e^{5x} = 6e^{5x}$	M1 A1		Substitution into differential equation
	$2k = 6 \Rightarrow k = 3$	A1ft	6	Only ft if xe^{5x} and x^2e^{5x} terms all cancel out
(b)	Aux. eqn. $m^2 - 10m + 25 = 0 \Rightarrow m = 5$	B1		PI
	CF is $(A + Bx)e^{5x}$	M1		
	GS $y = (A + Bx)e^{5x} + 3x^2e^{5x}$	M1 A1ft	4	Their CF + their/our PI ft only on wrong value of k
	Total		**10**	

5(a)	$u = \frac{dy}{dx} + x \Rightarrow \frac{du}{dx} = \frac{d^2y}{dx^2} + 1$	M1A1		
	$(x^2 - 1)\left(\frac{du}{dx} - 1\right) - 2x(u - x) = x^2 + 1$	M1		Substitution into LHS of DE as far as no *y*s
	DE $\Rightarrow (x^2 - 1)\frac{du}{dx} - 2xu = 0$			
	$\Rightarrow \frac{du}{dx} = \frac{2xu}{x^2 - 1}$	A1	4	CSO; AG
(b)	$\int \frac{1}{u}\, du = \int \frac{2x}{x^2 - 1}\, dx$	M1 A1		Separate variables
	$\ln u = \ln\lvert x^2 - 1\rvert + \ln A$	A1A1		
	$u = A(x^2 - 1)$	A1	5	
(c)	$\frac{dy}{dx} + x = A(x^2 - 1)$	M1		Use (b) $(\neq 0)$ to form DE in y and x
	$\frac{dy}{dx} = A(x^2 - 1) - x$			
	$y = A\left(\frac{x^3}{3} - x\right) - \frac{x^2}{2} + B$	M1 A1ft	3	Solution must have two different constants and correct method used to solve the DE
	Total		**12**	