FP3 Series & Limits Answers

2(a)	$\int xe^{-2x}\,dx = -\frac{1}{2}xe^{-2x} - \int -\frac{1}{2}e^{-2x}\,dx$	M1 A1		Reasonable attempt at parts
	$= -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\ \{+c\}$	A1√		Condone absence of $+c$
	$\int_0^a xe^{-2x}\,dx = -\frac{1}{2}ae^{-2a} - \frac{1}{4}e^{-2a} - (0 - \frac{1}{4})$	M1		$F(a) - F(0)$
	$= \frac{1}{4} - \frac{1}{2}ae^{-2a} - \frac{1}{4}e^{-2a}$	A1	5	
(b)	$\lim_{a \to \infty} a^k e^{-2a} = 0$	B1	1	
(c)	$\int_0^\infty xe^{-2x}\,dx =$			
	$= \lim_{a \to \infty} \{\frac{1}{4} - \frac{1}{2}ae^{-2a} - \frac{1}{4}e^{-2a}\}$	M1		If this line oe is missing then 0/2
	$= \frac{1}{4} - 0 - 0 = \frac{1}{4}$	A1√	2	On candidate's "1/4" in part (a). B1 must have been earned
	Total		**8**	

4(a)	$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 \ldots$	B1	1	
(b)(i)	$f(x) = e^{\sin x} \Rightarrow f(0) = 1$	B1		
	$f'(x) = \cos x\, e^{\sin x}$ $\Rightarrow f'(0) = 1$	M1A1		
	$f''(x) = -\sin x\, e^{\sin x} + \cos^2 x\, e^{\sin x}$ $f''(0) = 1$	M1A1		Product rule used
	Maclaurin $f(x) = f(0) + xf'(0) + \frac{x^2}{2}f''(0)$			
	so 1st three terms are $1 + x + \frac{1}{2}x^2$	A1	6	**CSO AG**
(ii)	$f'''(x) = \cos x(\cos^2 x - \sin x)\,e^{\sin x} +$ $+\{2\cos x(-\sin x) - \cos x\}\,e^{\sin x}$	M1A1		
	$f'''(0) = 0$ so the coefficient of x^3 in the series is zero	A1	3	**CSO AG** **SC** for (b): Use of series

				expansions….max of 4/9
(c)	$\sin x \approx x$.	B1		Ignore higher power terms in sin$x$ expansion
	$\dfrac{e^{\sin x}-1+\ln(1-x)}{x^2\sin x}=\dfrac{-\frac{1}{3}x^3+o(x^4)}{x^3}$	M1 A1		Series from (a) & (b) used Numerator kx^3 (+…)
	$=\dfrac{-\frac{1}{3}+o(x)}{1+o(x^2)}$			Condone if this step is missing
	$\lim_{x\to 0}\dfrac{e^{\sin x}-1+\ln(1-x)}{x^2\sin x}=-\dfrac{1}{3}$	A1√	4	On candidate's x^3 coefficient in (a) provided lower powers cancel
	Total		**14**	

5(a)	$\Rightarrow \lim_{a\to\infty}\left(\dfrac{3+\frac{2}{a}}{2+\frac{3}{a}}\right)=\dfrac{3+0}{2+0}=\dfrac{3}{2}$	M1 A1	2	
(b)	$\int_1^\infty \dfrac{3}{(3x+2)}-\dfrac{2}{2x+3}\,dx$			
	$=\left[\ln(3x+2)-\ln(2x+3)\right]_1^\infty$	M1 A1		$a\ln(3x+2)+b\ln(2x+3)$
	$=\left[\ln\left(\dfrac{3x+2}{2x+3}\right)\right]_1^\infty$	m1		
	$=\ln\left\{\lim_{a\to\infty}\left(\dfrac{3a+2}{2a+3}\right)\right\}-\ln 1$	M1		
	$=\ln\dfrac{3}{2}-\ln 1=\ln\dfrac{3}{2}$	A1	5	CSO
	Total		**7**	

7(a)(i)	$(1+y)^{-1} = 1 - y + y^2 ......$	B1	1	
(ii)	$\sec x \approx \dfrac{1}{1-\dfrac{x^2}{2}+\dfrac{x^4}{24}....}$	B1		
	$= \left[1-\dfrac{x^2}{2}+\dfrac{x^4}{24}....\right]^{-1} =$	M1		
	$\left\{1-\left(-\dfrac{x^2}{2}+\dfrac{x^4}{24}\right)+\left(-\dfrac{x^2}{2}+\dfrac{x^4}{24}\right)^2\right\}$	M1		
	$= \left\{1+\dfrac{x^2}{2}-\dfrac{x^4}{24}+\dfrac{x^4}{4}+...\right\}$			
	$= 1+\dfrac{x^2}{2}; + \dfrac{5x^4}{24}$	A1;A1	5	AG be convinced
	Alternative: Those using Maclaurin			
	$f(x) = \sec x$			
	$f(0) = 1$; $\underline{f'(x) = \sec x \tan x}$; $\{f'(0) = 0\}$	(B1)		
	$f''(x) = \sec x \tan^2 x + \sec^3 x$; $f''(0) = 1$	(M1)		Product rule oe
	$f'''(x) = \sec x \tan^3 x + 5\tan x \sec^3 x$;	(m1)		Chain rule with product rule OE
	$f^{(iv)}(x) = \sec x \tan^4 x + 18\tan^2 x \sec^3 x$ … $+5\sec^5 x \Rightarrow f^{(iv)}(0) = 5$			
	$\sec x \approx$ printed result	(A2)		CSO AG
(b)	$f(x) = \tan x$;			
	$f(0) = 0$; $f'(x) = \sec^2 x$; $\{f'(0) = 1\}$	B1		
	$f''(x) = 2\sec x(\sec x \tan x)$; $f''(0) = 0$			
	$f'''(x) = 4\sec x \tan x(\sec x \tan x) + 2\sec^4 x$	M1		Chain rule with product rule oe
	$f'''(0) = 2$			
	$\tan x = 0 + 1x + 0x^2 + \dfrac{2}{3!}x^3 … = x + \dfrac{1}{3}x^3$	A1	3	CSO AG
	Alternative: Those using otherwise			
	$.. = \dfrac{\sin x}{\cos x} \approx \left(x-\dfrac{x^3}{6}...\right)\left(1+\dfrac{x^2}{2}...\right)$	(M1) (A1)		
	$= x+\dfrac{x^3}{2}-\dfrac{x^3}{6}... = x+\dfrac{1}{3}x^3....$	(A1)		
(c)	$\left(\dfrac{x\tan 2x}{\sec x - 1}\right) = \dfrac{x\left(2x+o(x^3)\right)}{\dfrac{x^2}{2}+o(x^4)}$	B1 M1		$\tan 2x = 2x + \dfrac{1}{3}(2x)^3$ Condone $o(x^k)$ missing
	$= \dfrac{2+o(x^2)}{\dfrac{1}{2}+o(x^2)}$	M1		
	$\lim_{x\to 0}\left(\dfrac{x\tan 2x}{\sec x - 1}\right) = 4$	A1√	4	ft on $2k$ after B0 for $\tan 2x = kx+…$
	Total		**13**	

4(a)	Integrand is not defined at $x = 0$	E1	1	OE
(b)	$\int x^{-\frac{1}{2}} \ln x \, dx = 2x^{\frac{1}{2}} \ln x - \int 2x^{\frac{1}{2}} \left(\frac{1}{x}\right) dx$	M1		$... = kx^{\frac{1}{2}} \ln x \pm \int f(x)$, with $f(x)$ not involving the 'original' $\ln x$
		A1		
	$\ldots\ldots = 2x^{\frac{1}{2}} \ln x - 4x^{\frac{1}{2}} (+c)$	A1	3	Condone absence of '$+ c$'
(c)	$\int_0^e \frac{\ln x}{\sqrt{x}} \, dx = \lim_{a \to 0} \int_a^e \frac{\ln x}{\sqrt{x}} \, dx$	M1		
	$= -2e^{\frac{1}{2}} - \lim_{a \to 0} \left[2a^{\frac{1}{2}} \ln a - 4a^{\frac{1}{2}}\right]$	M1		$F(b) - F(a)$
	But $\lim_{a \to 0} a^{\frac{1}{2}} \ln a = 0$	B1		Accept a general form e.g. $\lim_{x \to 0} x^k \ln x = 0$
	So $\int_0^e \frac{\ln x}{\sqrt{x}} \, dx$ exists and $= -2e^{\frac{1}{2}}$	A1	4	
	Total		**8**	

6(a)(i)	$f'(x) = \frac{1}{2}(1+2x)^{-\frac{1}{2}}(2) = (1+2x)^{-\frac{1}{2}}$	M1A1		
	$f''(x) = -(1+2x)^{-\frac{3}{2}}$	A1F		ft a slip
	$f'''(x) = 3(1+2x)^{-\frac{5}{2}}$	A1	4	
(ii)	$f(x) = (1+2x)^{\frac{1}{2}} \Rightarrow f(0) = 1;$	B1		
	$f'(0) = 1;\ f''(0) = -1;\ f'''(0) = 3$	M1		All three attempted
		A1F		ft on $k(1+2x)'''$
	$f(x) = f(0) + xf'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{6} f'''(0)$			
	$\ldots \approx 1 + x - \frac{x^2}{2} + \frac{x^3}{2}$	A1	4	CSO **AG**
(b)	$e^x (1+2x)^{\frac{1}{2}} \approx$			
	$\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right)\left(1 + x - \frac{x^2}{2} + \frac{x^3}{2}\right)$	M1		Attempt to expand needed
	$\approx 1 + x(1+1) + x^2(-0.5 + 1 + 0.5)$ $+ x^3\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{2} + \frac{1}{6}\right)$	A1		

	$\approx 1+2x+x^2+\frac{2}{3}x^3$	A1	3	CSO
(c)	$e^{2x}=1+2x+\frac{(2x)^2}{2}+\frac{(2x)^3}{6}+\ldots$	B1	1	
	$=1+2x+2x^2+\frac{4}{3}x^3+\ldots$			
(d)	$1-\cos x=\frac{1}{2}x^2+\{o(x^4)\}$	B1		
	$\frac{e^x(1+2x)^{\frac{1}{2}}-e^{2x}}{1-\cos x}=$			
	$\frac{1+2x+x^2+\frac{2}{3}x^3-\left[1+2x+2x^2+\frac{4}{3}x^3\right]}{\frac{1}{2}x^2+\{o(x^4)\}}$	M1		Series used
	$\lim_{x\to 0}\ldots\ldots=\lim_{x\to 0}\frac{-x^2+\{o(x^3)\}}{\frac{1}{2}x^2+\{o(x^4)\}}=$	A1F		
	$\lim_{x\to 0}\frac{-1+o(x)}{\frac{1}{2}+o(x^2)}=-2$	A1F	4	ft a slip but must see the intermediate stage
	Total		**16**	

6(a)(i)	$f(x)=\ln(1+e^x)$:			
	$f(0)=\ln 2$	B1		
	$f'(x)=\frac{e^x}{1+e^x}\qquad f'(0)=\frac{1}{2}$	M1 A1		Chain rule
	$f''(x)=\frac{(1+e^x)e^x-e^xe^x}{(1+e^x)^2}=\frac{e^x}{(1+e^x)^2}$	M1 A1		Quotient rule OE
	$f''(0)=\frac{1}{4}$			
	so first three terms are:			
	$f(x)=\ln 2+\frac{1}{2}x+\frac{1}{4}\frac{x^2}{2!}=\ln 2+\frac{1}{2}x+\frac{1}{8}x^2$	A1	6	CSO; AG
(ii)	$f'''(x)=\frac{(1+e^x)^2e^x-e^x\left[2(1+e^x)e^x\right]}{(1+e^x)^4}$	M1 A1ft		Chain rule with quotient/product rule ft on $f''(x)=ke^x(1+e^x)^n$ (integer $n<0$)
	$f'''(0)=\frac{4-4}{2^4}=0$ {so coefficient of x^3 is zero}	A1	3	CSO; AG; All previous differentiation correct

SC for those not using Maclaurin's theorem: **maximum** of 4/9

(b)	$\frac{1}{2}x+\frac{1}{8}x^2$	B1	1	
(c)	$\ln\left(1-\frac{x}{2}\right)=$			
	$\left(-\frac{x}{2}\right)-\frac{1}{2}\left(-\frac{x}{2}\right)^2+\frac{1}{3}\left(-\frac{x}{2}\right)^3-\ldots\ldots$	B1	1	
(d)	$\ln\left(\frac{1+e^x}{2}\right)+\ln\left(1-\frac{x}{2}\right)=-\frac{x^3}{24}+\ldots$	M1		Uses previous expansions to obtain first non-zero term of the form kx^3
	$x-\sin x\approx x-\left[x-\frac{x^3}{3!}+\ldots\right]\approx\frac{x^3}{3!}+\ldots$	B1		
	$\left[\frac{\ln\left(\frac{1+e^x}{2}\right)+\ln\left(1-\frac{x}{2}\right)}{x-\sin x}\right]=\frac{-\frac{1}{24}x^3+\ldots}{\frac{1}{6}x^3+o\left(x^5\right)}$	M1		
	$=\frac{-\frac{1}{24}x^3+\ldots}{x^3\left[\frac{1}{6}+o\left(x^2\right)\right]}=\frac{-\frac{1}{24}+\ldots}{\frac{1}{6}+o\left(x^2\right)}$			
	$\lim\limits_{x\to 0}\ldots\ldots=\;-\frac{1}{4}$	A1	4	CSO
	Total		**15**	

7(a)	0	B1	1	
(b)	$u=xe^{-x}+1\Rightarrow du=\left(e^{-x}-xe^{-x}\right)dx$	M1		Attempts to find du
	$\int\frac{e^{-x}(1-x)}{xe^{-x}+1}\,dx=\int\frac{1}{u}\,du=\ln u+c$			
	$=\ln\left(xe^{-x}+1\right)\{+c\}$	A1	2	Condone missing c
(c)	$\int\frac{1-x}{x+e^x}\,dx=\int\frac{e^{-x}(1-x)}{xe^{-x}+1}dx$	B1		
	$\int_1^{\infty}\frac{1-x}{x+e^x}\,dx=\lim\limits_{a\to\infty}\left[\ln(xe^{-x}+1)\right]_1^a$			
	$=\lim\limits_{a\to\infty}\left\{\ln\left(ae^{-a}+1\right)\right\}-\ln\left(e^{-1}+1\right)$	M1		For using part (b) and $F(B)-F(A)$
	$=\ln\left\{\lim\limits_{a\to\infty}\left(ae^{-a}+1\right)\right\}-\ln\left(e^{-1}+1\right)$			
	$=\ln 1-\ln\left(e^{-1}+1\right)=-\ln\left(e^{-1}+1\right)$	M1 A1	4	For using limiting process
	Total		**7**	