5. (*a*) Given that $y = \ln [t + \sqrt{(1 + t^2)}]$, show that $\dfrac{dy}{dt} = \dfrac{1}{\sqrt{(1+t^2)}}$.

(3)

The curve C has parametric equations

$$x = \frac{1}{\sqrt{(1+t^2)}}, \quad y = \ln [t + \sqrt{(1 + t^2)}], \quad t \in \mathbb{R}.$$

A student was asked to prove that, for $t > 0$, the gradient of the tangent to C is negative.

The attempted proof was as follows:

$$y = \ln\left(t + \frac{1}{x}\right)$$

$$= \ln\left(\frac{tx+1}{x}\right)$$

$$= \ln (tx + 1) - \ln x$$

$$\therefore \frac{dy}{dx} = \frac{t}{tx+1} - \frac{1}{x}$$

$$= \frac{\frac{t}{x}}{t + \frac{1}{x}} - \frac{1}{x}$$

$$= \frac{t\sqrt{(1+t^2)}}{t+\sqrt{(1+t^2)}} - \sqrt{(1 + t^2)}$$

$$= -\frac{(1+t^2)}{t+\sqrt{(1+t^2)}}$$

As $(1 + t^2) > 0$, and $t + \sqrt{(1 + t^2)} > 0$ for $t > 0$, $\dfrac{dy}{dx} < 0$ for $t > 0$.

(*b*) (i) Identify the error in this attempt.

(ii) Give a correct version of the proof.

(6)

(*c*) Prove that $\ln [-t + \sqrt{(1 + t^2)}] = -\ln [t + \sqrt{(1 + t^2)}]$.

(3)

(*d*) Deduce that C is symmetric about the x-axis and sketch the graph of C.

(3)