4. A sequence of positive integers $a_1, a_2, a_3, \ldots$ has rth term given by

$$a_r = 2^r - 1$$

(a) Write down the first 6 terms of this sequence.

(1)

(b) Verify that $a_{r+1} = 2a_r + 1$

(1)

(c) Find $\sum_{r=1}^{n} a_r$

(3)

(d) Show that $\frac{1}{a_{r+1}} < \frac{1}{2} \times \frac{1}{a_r}$

(1)

(e) Hence show that $1 + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \frac{1}{31} + \ldots < 1 + \frac{1}{3} + \left(\frac{1}{7} + \frac{\frac{1}{2}}{7} + \frac{\frac{1}{4}}{7} + \ldots \right)$

(2)

(f) Show that $\frac{31}{21} < \sum_{r=1}^{\infty} \frac{1}{a_r} < \frac{34}{21}$

(5)