7.

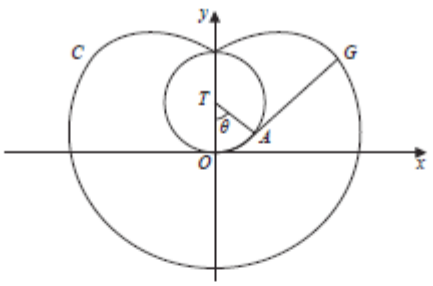

Figure 2

A circular tower stands in a large horizontal field of grass. A goat is attached to one end of a string and the other end of the string is attached to the fixed point O at the base of the tower. Taking the point O as the origin (0, 0), the centre of the base of the tower is at the point $T(0, 1)$. The radius of the base of the tower is 1. The string has length π and you may ignore the size of the goat. The curve C represents the edge of the region that the goat can reach as shown in Figure 2.

(a) Write down the equation of C for $y < 0$.

(1)

When the goat is at the point $G(x, y)$, with $x > 0$ and $y > 0$, as shown in Figure 2, the string lies along OAG where OA is an arc of the circle with angle $OTA = \theta$ radians and AG is a tangent to the circle at A.

(b) With the aid of a suitable diagram show that

$$x = \sin\theta + (\pi - \theta)\cos\theta$$

$$y = 1 - \cos\theta + (\pi - \theta)\sin\theta$$

(5)

(c) By considering $\int y \frac{dx}{d\theta}\, d\theta$, show that the area between C, the positive x-axis and the positive y-axis can be expressed in the form

$$\int_0^{\pi} u \sin u \, du + \int_0^{\pi} u^2 \sin^2 u \, du + \int_0^{\pi} u \sin u \cos u \, du$$

(5)

(d) Show that $\int_0^{\pi} u^2 \sin^2 u \, du = \frac{\pi^3}{6} + \int_0^{\pi} u \sin u \cos u \, du$

(4)

(e) Hence find the area of grass that can be reached by the goat.

(8)