(i) Show, with working, that

$$x^3 - (1 + \cos\theta + \sin\theta)\, x^2 + (\cos\theta \sin\theta + \cos\theta + \sin\theta)\, x - \sin\theta \cos\theta, \qquad (1)$$

equals

$$(x - 1)\left(x^2 - (\cos\theta + \sin\theta)\, x + \cos\theta \sin\theta\right)$$

Deduce that the cubic in (1) has roots

$$1, \qquad \cos\theta, \qquad \sin\theta.$$

(ii) Give the roots when $\theta = \frac{\pi}{3}$.

(iii) Find all values of θ in the range $0 \leqslant \theta < 2\pi$ such that two of the three roots are equal.

(iv) What is the greatest possible difference between two of the roots, and for what values of θ in the range $0 \leqslant \theta < 2\pi$ does this greatest difference occur?

Show that for each such θ the cubic (1) is the same.