

Songs of the Martian classical period had just two notes (let us call them x and y) and were constructed according to rigorous rules:

- I. the sequence consisting of no notes was deemed to be a song (perhaps the most pleasant);
- II. a sequence starting with x , followed by two repetitions of an existing song and ending with y was also a song;
- III. the sequence of notes obtained by interchanging x s and y s in a song was also a song.

All songs were constructed using those rules.

- (i) Write down four songs of length six (that is, songs with exactly six notes).
- (ii) Show that if there are k songs of length m then there are $2k$ songs of length $2m + 2$. Deduce that for each natural number there are 2^n songs of length $2^{n+1} - 2$.

Songs of the Martian later period were constructed using also the rule:

- IV. if a song ended in y then the sequence of notes obtained by omitting that y was also a song.
- (iii) What lengths do songs of the later period have? That is, for which natural numbers n is there a song with exactly n notes? Justify your answer.