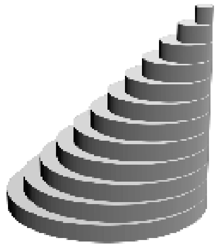
UKMT Circles Questions
(Answers follow after all the questions)

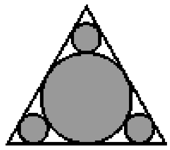
2005…

11. A sculpture is made up of 12 wooden cylinders, each of height 2cm. They are glued together as shown. The diameter of the top cylinder is 2cm and each of the other cylinders has a diameter 2cm more than the one immediately above it. The exhibit stands with its base on a marble table. What, in cm^2, is the total surface area of the sculpture, excluding the base?



A 456π B 356π C 256π D 156π E 144π

23. The diagram shows four touching circles, each of which also touches the sides of an equilateral triangle with sides of length 3.
What is the area of the shaded region?



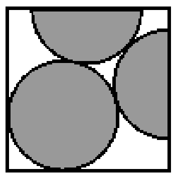
A $\dfrac{11\pi}{12}$ B π C $\dfrac{(4+\sqrt{3})\pi}{6}$ D $\dfrac{(3+\sqrt{3})\pi}{4}$ E $\dfrac{37\pi}{36}$

2006…

9. The 80 spokes of the giant wheel *The London Eye* are made from 4 miles of cable. Roughly what is the circumference of the wheel in metres?

A 50 B 100 C 500 D 750 E 900

23. In the diagram, the circle and the two semicircles have radius 1.
What is the perimeter of the square?

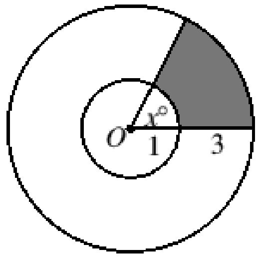


A $6 + 4\sqrt{2}$ B $2 + 4\sqrt{2} + 2\sqrt{3}$ C $3\sqrt{2} + 4\sqrt{3}$ D $4 + 2\sqrt{2} + 2\sqrt{6}$ E 12

2007…

14. The point O is the centre of both circles and the shaded area is one-sixth of the area of the outer circle.
What is the value of x?

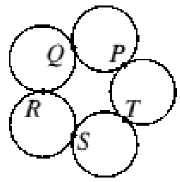
A 60 B 64 C 72 D 80 E 84



2008…

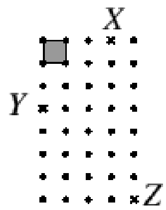
14. Five touching circles each have radius 1 and their centres are at the vertices of a regular pentagon. What is the radius of the circle through the points of contact P, Q, R, S and T?

A $\tan 18°$ B $\tan 36°$ C $\tan 45°$ D $\tan 54°$ E $\tan 72°$



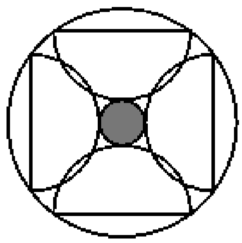
18. The shaded square of the lattice shown has area 1. What is the area of the circle through the points X, Y and Z?

A $\frac{9\pi}{2}$ B 8π C $\frac{25\pi}{2}$ D 25π E 50π



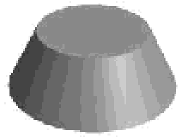
20. The diagram shows four semicircles symmetrically placed between two circles. The shaded circle has area 4 and each semicircle has area 18. What is the area of the outer circle?

A $72\sqrt{2}$ B 100 C 98 D 96 E $32\sqrt{3}$



2009…

21. A frustum is the solid obtained by slicing a right-circular cone perpendicular to its axis and removing the small cone above the slice. This leaves a shape with two circular faces and a curved surface. The original cone has base radius 6 cm and height 8 cm, and the curved surface area of the frustum is equal to the area of the two circles. What is the height of the frustum?



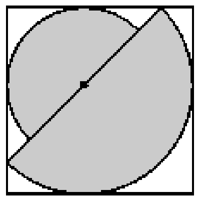
A 3 cm B 4 cm C 5 cm D 6 cm E 7 cm

2010…

16. $PQRS$ is a quadrilateral inscribed in a circle of which PR is a diameter. The lengths of PQ, QR and RS are 60, 25 and 52 respectively. What is the length of SP?

A $21\frac{2}{3}$ B $28\frac{11}{13}$ C 33 D 36 E 39

23. The diagram shows two different semicircles inside a square with sides of length 2. The common centre of the semicircles lies on a diagonal of the square.
What is the total shaded area?

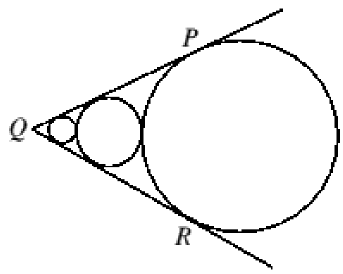


A π B $6\pi(3-2\sqrt{2})$ C $\pi\sqrt{2}$ D $3\pi(2-\sqrt{2})$ E $8\pi(2\sqrt{2}-3)$

2011…

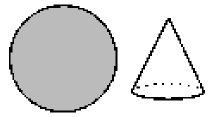
24. Three circles and the lines PQ and QR touch as shown. The distance between the centres of the smallest and the biggest circles is 16 times the radius of the smallest circle. What is the size of $\angle PQR$?

A 45° B 60° C 75° D 90° E 135°



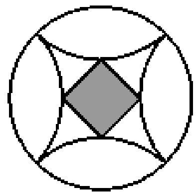
2012…

11. Coco is making clown hats from a circular piece of cardboard. The circumference of the base of each hat equals its slant height, which in turn is equal to the radius of the piece of cardboard. What is the maximum number of hats that Coco can make from the piece of cardboard?



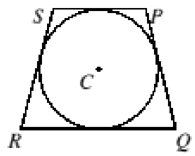
A 3 B 4 C 5 D 6 E 7

17. The diagram shows a pattern found on a floor tile in the cathedral in Spoleto, Umbria. A circle of radius 1 surrounds four quarter circles, also of radius 1, which enclose a square. The pattern has four axes of symmetry. What is the side length of the square?



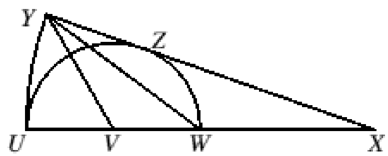
A $\frac{1}{\sqrt{2}}$ B $2-\sqrt{2}$ C $\frac{1}{\sqrt{3}}$ D $\frac{1}{2}$ E $\sqrt{2}-1$

20. In trapezium $PQRS$, $SR = PQ = 25$cm and SP is parallel to RQ. All four sides of $PQRS$ are tangent to a circle with centre C. The area of the trapezium is 600cm^2. What is the radius of the circle?



A 7.5cm B 8cm C 9cm D 10cm E 12cm

22. A semicircle of radius r is drawn with centre V and diameter UW. The line UW is then extended to the point X, such that UW and WX are of equal length. An arc of the circle with centre X and radius $4r$ is then drawn so that the line XY is a tangent to the

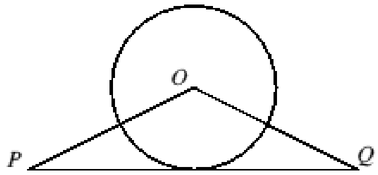


semicircle at Z, as shown. What, in terms of r, is the area of triangle YVW?

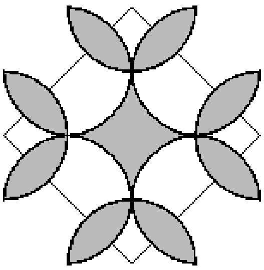
A $\frac{4r^2}{9}$ B $\frac{2r^2}{3}$ C r^2 D $\frac{4r^2}{3}$ E $2r^2$

2013…

11. The diagram shows a circle with centre O and a triangle OPQ. Side PQ is a tangent to the circle. The area of the circle is equal to the area of the triangle. What is the ratio of the length of PQ to the circumference of the circle?



A $1:1$ B $2:3$ C $2:\pi$ D $3:2$ E $\pi:2$

21. 

The shaded design shown in the diagram is made by drawing eight circular arcs, all with the same radius. The centres of four arcs are the vertices of the square; the centres of the four touching arcs are the midpoints of the sides of the square. The diagonals of the square have length 1.

What is the total length of the border of the shaded design?

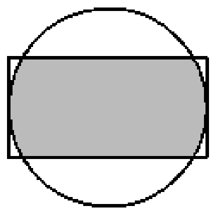
A 2π B $\frac{5\pi}{2}$ C 3π D $\frac{7\pi}{2}$ E 4π

2014…

16. The diagram shows a rectangle measuring 6×12 and a circle.

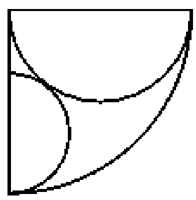
The two shorter sides of the rectangle are tangents to the circle. The circle and rectangle have the same centre.

The region that lies inside both the rectangle and the circle is shaded. What is its area?



A $12\pi + 18\sqrt{3}$ B $24\pi - 3\sqrt{3}$ C $18\pi - 8\sqrt{3}$ D $18\pi + 12\sqrt{3}$ E $24\pi + 18\sqrt{3}$

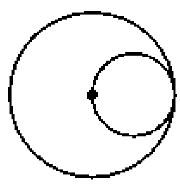
19. The diagram shows a quadrant of radius 2, and two touching semicircles. The larger semicircle has radius 1. What is the radius of the smaller semicircle?



A $\frac{\pi}{6}$ B $\frac{\sqrt{3}}{2}$ C $\frac{1}{2}$ D $\frac{1}{\sqrt{3}}$ E $\frac{2}{3}$

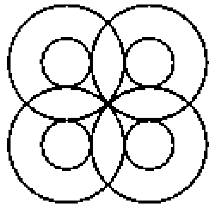
2015…

4. In the diagram, the smaller circle touches the larger circle and also passes through its centre. What fraction of the area of the larger circle is outside the smaller circle?



A $\frac{2}{3}$ B $\frac{3}{4}$ C $\frac{4}{5}$ D $\frac{5}{6}$ E $\frac{6}{7}$

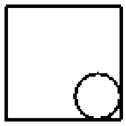
17. The diagram shows eight circles of two different sizes. The circles are arranged in concentric pairs so that the centres form a square. Each larger circle touches one other larger circle and two smaller circles. The larger circles have radius 1. What is the radius of each smaller circle?



A $\frac{1}{3}$ B $\frac{2}{5}$ C $\sqrt{2}-1$ D $\frac{1}{2}$ E $\frac{1}{2}\sqrt{2}$

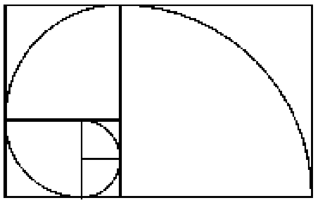
2016…

3. The diagram shows a circle with radius 1 that rolls without slipping around the inside of a square with sides of length 5.
The circle rolls once around the square, returning to its starting point.
What distance does the centre of the circle travel?

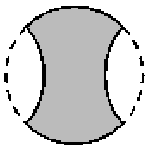


A $16-2\pi$ B 12 C $6+\pi$ D $20-2\pi$ E 20

13. Five square tiles are put together side by side. A quarter circle is drawn on each tile to make a continuous curve as shown. Each of the smallest squares has side-length 1.
What is the total length of the curve?
A 6π B 6.5π C 7π D 7.5π E 8π



18. The circumference of a circle with radius 1 is divided into four equal arcs. Two of the arcs are 'turned over' as shown.
What is the area of the shaded region?



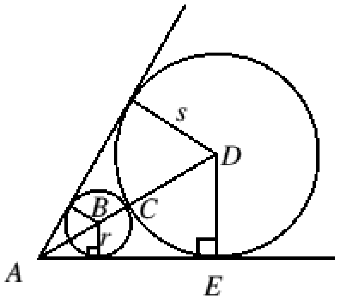
A 1 B $\sqrt{2}$ C $\frac{1}{2}\pi$ D $\sqrt{3}$ E 2

UKMT Circles Answers

2005…

11. A The diameter of the largest cylinder is 24cm, so the sum of the areas of the horizontal parts of the sculpture, excluding its base, is that of a circle of diameter 24cm, that is $144\pi\ \text{cm}^2$. The sum of the areas of the vertical parts of the sculpture is $(2\pi \times 1 \times 2 + 2\pi \times 2 \times 2 + 2\pi \times 3 \times 2 + \ldots + 2\pi \times 12 \times 2)\ \text{cm}^2$, that is $312\pi\ \text{cm}^2$. So, excluding the base, the total surface area of the sculpture is $456\pi\ \text{cm}^2$.

23. B We note from the symmetry of the figure that the three small circles have the same radius. Let this be r and let the radius of the large circle be s. Let A, B, C, D, E be the points shown on the diagram.

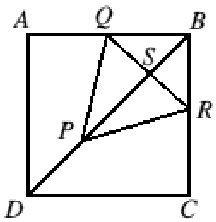


By symmetry, $\angle DAE = 30°$.
Now $\frac{DE}{AD} = \sin 30° = \frac{1}{2}$ so AD has length $2s$. Similarly, AB has length $2r$.
Since $AD = AB + BC + CD$, the length of AD is also given by $2r + r + s$. Hence $2s = 3r + s$, i.e. $s = 3r$.
Also, $\frac{DE}{AE} = \frac{s}{s/2} = \tan 30° = \frac{1}{\sqrt{3}}$ so $s = \frac{3}{2\sqrt{3}}$. Hence $r = \frac{1}{2\sqrt{3}}$.
Thus the shaded area $= \pi s^2 + 3\pi r^2 = \pi \times \frac{9}{12} + 3\pi \times \frac{1}{12} = \pi$.

2006…

9. C Each spoke of the London Eye is about $\frac{1}{20}$ mile long. As 1 mile is approximately 1600 m, this means that the radius of the giant wheel is roughly 80 m. So the circumference is approximately 500 m.

23. D Let the vertices of the square be A, B, C, D and the centres of the circle and the two semicircles be P, Q, R, as shown. The midpoint of QR is S. By symmetry, P and S both lie on diagonal BD of square $ABCD$ and the whole figure is symmetrical about BD.



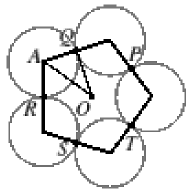
As P is distance 1 from both AD and DC, the length of DP is $\sqrt{2}$.
As the circles and semicircles are mutually tangent, PQR is an equilateral triangle of side 2, so the length of PS is $\sqrt{3}$. As angles QBS and BSQ are 45° and 90° respectively, triangle SBQ is isosceles, so $SB = SQ = 1$. Hence the length of BD is $\sqrt{2} + \sqrt{3} + 1$. Now the length of the side of the square is $BD \div \sqrt{2}$ so the perimeter of the square is $4 \times (BD \div \sqrt{2})$, that is $2\sqrt{2} \times BD$.
So the perimeter is $2\sqrt{2}(\sqrt{2} + \sqrt{3} + 1)$, that is $4 + 2\sqrt{6} + 2\sqrt{2}$.

2007…

14. B The shaded area is $\frac{x}{360}(\pi \times 4^2 - \pi \times 1^2) = \frac{15\pi x}{360} = \frac{\pi x}{24}$. So $\frac{\pi x}{24} = \frac{\pi \times 4^2}{6}$; thus $x = 64$.

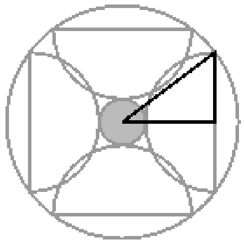
2008…

14. D The internal angle of a regular pentagon is 108°. Let A be the centre of a touching circle, as shown. Since OA bisects $\angle RAQ$, $\angle OAQ = 54°$. Also, triangle OAQ is right-angled at Q (radius perpendicular to tangent). Since $AQ = 1$, $OQ = \tan 54°$.



18. C Since $XY^2 = 18$, $YZ^2 = 32$ and $XZ^2 = 50$, we have $XZ^2 = XY^2 + YZ^2$. Hence by the converse of Pythagoras' Theorem, $\angle XYZ = 90°$. Since the angle in a semi-circle is 90° the segment XZ is the diameter of the specified circle. Hence the radius is $\frac{1}{2}\sqrt{50}$ and the area of the circle is $\frac{50\pi}{4} = \frac{25\pi}{2}$.

20. B Let r_1, r_2 and r_3 be the radii of the shaded circle, semicircles and outer circle respectively. A right-angled triangle can be formed with sides r_3, $(r_1 + r_2)$ and r_2.
Hence, by Pythagoras' Theorem, $r_3^2 = (r_1 + r_2)^2 + r_2^2$.
Now $\pi r_1^2 = 4$, hence $r_1 = 2/\sqrt{\pi}$. Likewise $r_2 = 6/\sqrt{\pi}$.
Hence $r_2 = 3r_1$ so that $r_3^2 = (r_1 + 3r_1)^2 + (3r_1)^2 = 25r_1^2$. Thus the required area is $25 \times 4 = 100$.



2009…

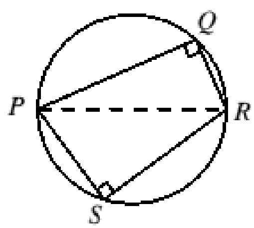
21. B Let r be the radius of the small cone and h the height.
Let l_1 and l_2 be the slant heights of the small and large cones respectively.
By Pythagoras' Theorem $l_2 = \sqrt{6^2 + 8^2} = 10$.
Using similar triangles, $\frac{l_1}{r} = \frac{10}{6}$ so $l_1 = \frac{5}{3}r$ and $\frac{h}{8} = \frac{r}{6}$ giving $h = \frac{4}{3}r$.
Thus the area of the curved surface of the frustum is

$$\pi \times 6 \times 10 - \pi \times r \times \frac{5}{3} \times r = \pi\left(60 - \frac{5r^2}{3}\right).$$

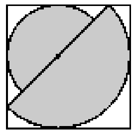
The sum of the areas of the two circles is $\pi \times 6^2 + \pi \times r^2 = \pi(36 + r^2)$.
Hence $\pi\left(60 - \frac{5r^2}{3}\right) = \pi(36 + r^2)$ and so $24 = \frac{8r^2}{3}$ giving $r = 3$, so $h = \frac{4}{3} \times 3 = 4$.
Therefore, in cms, the height of the frustum is $8 - 4 = 4$.

2010…

16. E As PR is a diameter, $\angle PQR = \angle PSR = 90°$ (angles in a semicircle are 90°).
Since $PQ = 12 \times 5$ and $QR = 5 \times 5$, triangle PQR is an enlarged 5, 12, 13 triangle and so $PR = 13 \times 5 = 65$.
Since $PR = 5 \times 13$ and $SR = 4 \times 13$, triangle PRS is an enlarged 3, 4, 5 triangle and so $SP = 3 \times 13 = 39$.



23. **B** Let r_1 and r_2 represent the radii of the smaller and larger semicircles respectively. A vertical line through the common centre of the semicircles gives $r_1 + r_2 = 2$ … (1). Also, together with the diameter of the larger semicircle, this line forms a right-angled, isosceles triangle giving $\sin 45° = \frac{r_1}{r_2}$. Hence $r_2 = \sqrt{2}r_1$ … (2).



Substituting (2) into (1) gives $(1 + \sqrt{2})r_1 = 2$ so that $r_1 = 2(\sqrt{2} - 1)$.
Therefore $r_2 = 2\sqrt{2}(\sqrt{2} - 1)$.
Hence the total shaded area is $\frac{1}{2}\pi(r_1^2 + r_2^2) = \frac{1}{2}\pi\left[4(\sqrt{2} - 1)^2 + 8(\sqrt{2} - 1)^2\right] = 6\pi(3 - 2\sqrt{2})$.

2011…

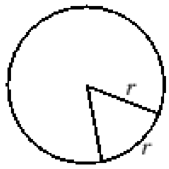
24. **B** Let the radii of the circles from smallest to largest be r_1, r_2 and r_3 respectively. Hence $16r_1 = r_3 + 2r_2 + r_1$, thus $r_3 = 15r_1 - 2r_2$… (1). Let $r_1 + x$ be the distance from Q to the centre of the smallest circle. By similar triangles,

$$\frac{r_1}{r_1 + x} = \frac{r_2}{x + 2r_1 + r_2} = \frac{r_3}{16r_1 + r_1 + x}\ldots (2).$$

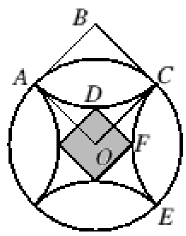
Thus $r_1(x + 2r_1 + r_2) = r_2(r_1 + x)$. Hence $r_2 = \frac{r_1x + 2r_1^2}{x}$… (3). From (1) and (2) $\frac{r_1x}{r_1 + x} = \frac{(15r_1 - 2r_2)x}{17r_1 + x}$ hence $\frac{r_1x}{r_1 + x} = \frac{15r_1x - 2(r_1x + 2r_1^2)}{17r_1 + x}$. Dividing throughout by r_1 and simplifying gives $12x^2 - 8r_1x - 4r_1^2 = 0$. Hence $(3x + r_1)(x - r_1) = 0$ so, as $r_1 > 0$, $x = r_1$. Thus $\sin\frac{\angle PQR}{2} = \frac{r_1}{r_1 + x} = \frac{r_1}{2r_1} = \frac{1}{2}$. Hence $\frac{1}{2}\angle PQR = 30°$ so $\angle PQR = 60°$.

2012…

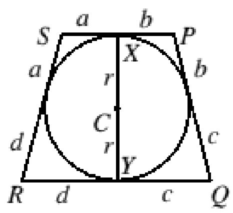
11. **D** Let the radius of the circular piece of cardboard be r. The diagram shows a sector of the circle which would make one hat, with the minor arc shown becoming the circumference of the base of the hat. The circumference of the circle is $2\pi r$. Now $6r < 2\pi r < 7r$. This shows that we can cut out 6 hats in this fashion and also shows that the area of cardboard unused in cutting out *any* 6 hats is less than the area of a single hat. Hence there is no possibility that more than 6 hats could be cut out.



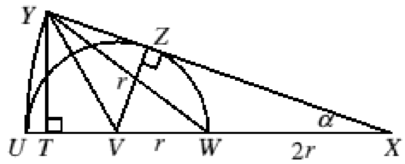
17. **B** In the diagram, B is the centre of the quarter-circle arc AC; D is the point where the central square touches arc AC; F is the point where the central square touches arc CE; O is the centre of the circle.
As both the circle and arc AC have radius 1, $OABC$ is a square of side 1.
By Pythagoras' Theorem: $OB^2 = 1^2 + 1^2$. So $OB = \sqrt{2}$. Therefore $OD = OB - DB = \sqrt{2} - 1$. By a similar argument, $OF = \sqrt{2} - 1$. Now $DF^2 = OD^2 + OF^2 = 2 \times OD^2$ since $OD = OF$. So the side of the square is $\sqrt{2} \times OD = \sqrt{2}(\sqrt{2} - 1) = 2 - \sqrt{2}$.



20. **E** The two tangents drawn from a point outside a circle to that circle are equal in length. This theorem has been used to mark four pairs of equal line segments on the diagram. In the circle the diameter, XY, has been marked. It is also a perpendicular height of the trapezium. We are given that $SR = PQ = 25$ cm so we can deduce that $(a + d) + (b + c) = 25 + 25 = 50$. The area of trapezium $PQRS = \frac{1}{2}(SP + QR) \times XY = 600\text{ cm}^2$. Therefore $\frac{1}{2}(a + b + c + d) \times 2r = 600$. So $\frac{1}{2} \times 50 \times 2r = 600$, i.e. $r = 12$.



22. **B** Let the perpendicular from Y meet UV at T and let $\angle ZXV = \alpha$. Note that $\angle VZX = 90°$ as a tangent to a circle is perpendicular to the radius at the point of contact. Therefore $\sin\alpha = \frac{r}{3r} = \frac{1}{3}$. Consider triangle YTX: $\sin\alpha = \frac{YT}{YX}$. So $YT = YX\sin\alpha = \frac{4r}{3}$. So the area of triangle $YVW = \frac{1}{2} \times VW \times YT = \frac{1}{2} \times r \times \frac{4r}{3} = \frac{2r^2}{3}$.



2013…

11. **A** Let the radius of the circle be r. Then its area is πr^2. The height of the triangle is r and its area is $\frac{1}{2} \times PQ \times r$. So $\frac{1}{2} \times PQ \times r = \pi r^2$ and therefore $PQ = 2\pi r$, which is also the circumference of the circle. Therefore the ratio of the length of PQ to the circumference of the circle is 1 : 1.

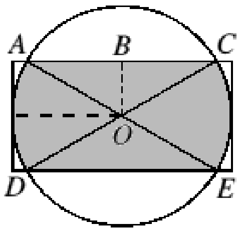
21. **B** Let the top vertex of the square be A and the midpoints of the two lines that meet at A be B and C. The line BC is of length $\frac{1}{2}$ and is perpendicular to the diagonal of the square through A. Let the point of intersection of these two lines be D. Let the end of the uppermost arc, above B, be E. Then $ADBE$ is a rhombus, made from four radii of the arcs, AD, DB, BE and EA, each of length $\frac{1}{4}$. As $\angle ADB = 90°$, this rhombus is a square.

It then follows that the four arcs whose centres are the vertices of the original square are all semi-circles. The remaining four touching arcs are each $\frac{3}{4}$ of a circle.

In total, the length of the border is $4 \times \frac{1}{2} + 4 \times \frac{3}{4}$ times the circumference of a circle with the same radius, so is $5 \times 2\pi \times \frac{1}{4} = \frac{5}{2}\pi$.

2014…

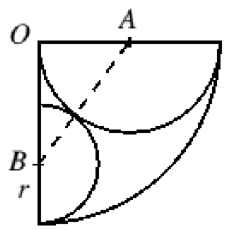
16. **A** The diameter of the circle is the same length as the longest sides of the rectangle, so the radius of the circle is 6. The perpendicular distance from the centre of the circle to the longest sides of the rectangle is half of the length of the shortest sides which is 3.



Drawing two diameters AE and DC as shown splits the shaded area into two sectors and two isosceles triangles. As OA is 6 and OB is 3, $\angle AOB = 60°$ and, by Pythagoras' Theorem, $AB = 3\sqrt{3}$. Thus $\angle AOD = 180° - 2 \times 60° = 60°$. So the shaded area is

$$2 \times \frac{60}{360} \times \pi \times 6^2 + 2 \times \frac{1}{2} \times 2 \times 3\sqrt{3} \times 3 = 12\pi + 18\sqrt{3}.$$

19. **E** Let the centre of the quadrant be O, the centre of the larger semicircle be A and the centre of the smaller semicircle be B. Let the radius of the smaller semicircle be r. It is given that $OA = 1$. The common tangent to the two semicircles at the point of contact makes an angle of 90° with the radius of each semicircle. Therefore the line AB passes through the point of contact, as $2 \times 90° = 180°$ and angles on a straight line sum to 180°. So the line AB has length $r + 1$. This is the hypotenuse of the right-angled triangle OAB in which $OA = 1$ and $OB = 2 - r$. By Pythagoras' Theorem $(2 - r)^2 + 1^2 = (r + 1)^2$, so $4 - 4r + r^2 + 1 = r^2 + 2r + 1$ and therefore $4 = 6r$ and so $r = \frac{2}{3}$.



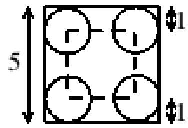
2015…

4. **B** Let the radius of the smaller circle be r and so the radius of the larger circle is $2r$. The area of the smaller circle is then πr^2 and the area of the larger circle is $\pi \times (2r)^2$ which is $4\pi r^2$. The fraction of the larger circle which is outside the smaller circle is then $\frac{4\pi r^2 - \pi r^2}{4\pi r^2} = \frac{3\pi r^2}{4\pi r^2} = \frac{3}{4}$.

17. **C** Let the radius of each of the smaller circles be r and let the centres of the circles be A, B, C and D in order. We are given that $ABCD$ is a square. When two circles touch externally, the distance between their centres equals the sum of their radii. Hence AB and BC have length $r + 1$ and AC has length $1 + 1 = 2$. By Pythagoras' Theorem $(r+1)^2 + (r+1)^2 = 2^2$, so $2(r+1)^2 = 2^2 = 4$ and therefore $(r+1)^2 = 2$. Square rooting both sides gives $r + 1 = \sqrt{2}$, as we must take the positive root, and so $r = \sqrt{2} - 1$.

2016…

3. **B** As the circle rolls, the centre of the circle moves along four straight lines shown as dashed lines. Each dashed line has length $5 - (1 + 1)$ so the total distance travelled is 4×3 which is 12.



13. **A** A quarter circle of radius r has length $\frac{2\pi r}{4}$ which is $\frac{\pi r}{2}$. The total length of the curve shown is then $\frac{\pi}{2}(1 + 1 + 2 + 3 + 5)$ which is 6π.

18. **E** The four arcs are of equal length and their end-points lie on a circle, so the four end-points can be joined to make a square. As two of the arcs are 'turned over', the two unshaded regions inside the square have areas equal to the two shaded regions outside the square. The total shaded area is therefore equal to the area of the square. The radius of the circle is given as 1 so, by Pythagoras' Theorem, the side-length of the square is $\sqrt{1^2 + 1^2} = \sqrt{2}$. So the area of the shaded region is $\sqrt{2} \times \sqrt{2} = 2$.