UKMT Combinations & Probability Questions

(Answers follow after all the questions)

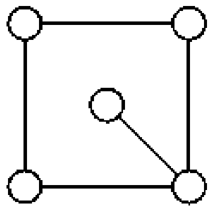
2005…

16. A hockey team consists of 1 goalkeeper, 4 defenders, 4 midfielders and 2 forwards. There are 4 substitutes: 1 goalkeeper, 1 defender, 1 midfielder and 1 forward. A substitute may only replace a player of the same category eg: midfielder for midfielder. Given that a maximum of 3 substitutes may be used and that there are still 11 players on the pitch at the end, how many different teams could finish the game?

A 110 B 118 C 121 D 125 E 132

2006…

13. The diagram shows five discs connected by five line segments.
Three colours are available to colour these discs.
In how many different ways is it possible to colour all five discs if discs which are connected by a line segment are to have different colours?



A 6 B 12 C 30 D 36 E 48

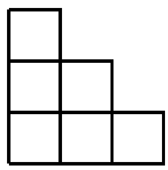
2007…

21. A bracelet is to be made by threading four identical red beads and four identical yellow beads onto a hoop. How many different bracelets can be made?

A 4 B 8 C 12 D 18 E 24

2008…

6. It is required to shade at least one of the six small squares in the diagram on the right so that the resulting figure has exactly one axis of symmetry. In how many different ways can this be done?



A 6 B 9 C 10 D 12 E 15

2009…

6. A bag contains hundreds of glass marbles, each one coloured either red, orange, green or blue. There are more than 2 marbles of each colour.
Marbles are drawn randomly from the bag, one at a time, and not replaced.
How many marbles must be drawn from the bag in order to ensure at least three marbles of the same colour are drawn?

A 4 B 7 C 9 D 12 E 13

2010…

7. There are 120 different arrangements of the five letters in the word ANGLE. If all 120 are listed in alphabetical order starting with AEGLN and finishing with NLGEA, which position in the list does ANGLE occupy?

A 18th B 20th C 22nd D 24th E 26th

20. There are 10 girls in a mixed class. If two pupils from the class are selected at random to represent the class on the School Council, then the probability that both are girls is 0.15. How many boys are in the class?

A 10 B 12 C 15 D 18 E 20

25. All the digits of a number are different, the first digit is not zero, and the sum of the digits is 36. There are $N \times 7!$ such numbers. What is the value of N?

A 72 B 97 C 104 D 107 E 128

2012…

14. Six students who share a house all speak exactly two languages. Helga speaks only English and German; Ina speaks only German and Spanish; Jean-Pierre speaks only French and Spanish; Karim speaks only German and French; Lionel speaks only French and English whilst Mary speaks only Spanish and English. If two of the students are chosen at random, what is the probability that they speak a common language?

A $\frac{1}{2}$ B $\frac{2}{3}$ C $\frac{3}{4}$ D $\frac{4}{5}$ E $\frac{5}{6}$

23. Tom and Geri have a competition. Initially, each player has one attempt at hitting a target. If one player hits the target and the other does not then the successful player wins. If both players hit the target, or if both players miss the target, then each has another attempt, with the same rules applying. If the probability of Tom hitting the target is always $\frac{4}{5}$ and the probability of Geri hitting the target is always $\frac{2}{3}$, what is the probability that Tom wins the competition?

A $\dfrac{4}{15}$ B $\dfrac{8}{15}$ C $\dfrac{2}{3}$ D $\dfrac{4}{5}$ E $\dfrac{13}{15}$

2013…

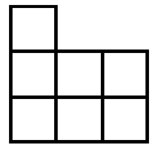
12. As a special treat, Sammy is allowed to eat five sweets from his very large jar which contains many sweets of each of three flavours – Lemon, Orange and Strawberry. He wants to eat his five sweets in such a way that no two consecutive sweets have the same flavour. In how many ways can he do this?

A 32 B 48 C 72 D 108 E 162

2014…

15. The figure shown alongside is made from seven small squares. Some of these squares are to be shaded so that:
    (i) at least two squares are shaded;
    (ii) two squares meeting along an edge or at a corner are not both shaded.

    How many ways are there to do this?



A 4 B 8 C 10 D 14 E 18

22. A bag contains m blue and n yellow marbles. One marble is selected at random from the bag and its colour is noted. It is then returned to the bag along with k other marbles of the same colour. A second marble is now selected at random from the bag. What is the probability that the second marble is blue?

A $\frac{m}{m+n}$ B $\frac{n}{m+n}$ C $\frac{m}{m+n+k}$ D $\frac{m+k}{m+n+k}$ E $\frac{m+n}{m+n+k}$

2015…

16. Fnargs are either red or blue and have 2, 3 or 4 heads. A group of six Fnargs consisting of one of each possible form is made to line up such that no immediate neighbours are the same colour nor have the same number of heads. How many ways are there of lining them up from left to right?

A 12 B 24 C 60 D 120 E 720

24. Peter has 25 cards, each printed with a different integer from 1 to 25. He wishes to place N cards in a single row so that the numbers on every adjacent pair of cards have a prime factor in common.
    What is the largest value of N for which this is possible?

A 16 B 18 C 20 D 22 E 24

2016…

17. Aaron has to choose a three-digit code for his bike lock. The digits can be chosen from 1 to 9. To help him remember them, Aaron chooses three different digits in increasing order, for example 278. How many such codes can be chosen?

A 779 B 504 C 168 D 84 E 9

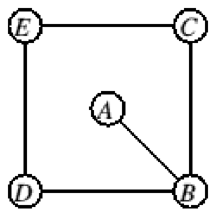
UKMT Combinations & Probability Answers

2005…

16. B Firstly, we note that of the players on the pitch at the end of the game, the goalkeeper is one of two players; the four defenders form one of five different possible combinations, as do the four midfielders, and the two forwards form one of three different possible combinations. So, if up to four substitutes were allowed, the number of different teams which could finish the game would be $2 \times 5 \times 5 \times 3$, that is 150.
From this number we must subtract the number of these teams which require four substitutions to be made. This is $1 \times 4 \times 4 \times 2$, that is 32, so the required number of teams is 118.

2006…

13. D Disc A may have any one of three colours and, for each of these, disc B may have two colours. So these two discs may be coloured in six different ways.



If discs C and D have the same colour, then they may be coloured in two different ways and, for each of these, disc E may have two colours. So the discs may be coloured in 24 different ways if C and D are the same colour. However, if discs C and D are different colours, then C may have one of two colours, but the colours of discs D and E are then determined. So the discs may be coloured in 12 different ways if C and D are different colours. In total, therefore, the discs may be coloured in 36 different ways.

2007…

21. B In this solution, the notation $p / q / r / s / \ldots$ represents p beads of one colour, followed by q beads of the other colour, followed by r beads of the first colour, followed by s beads of the second colour etc.
Since the colours alternate, there must be an even number of these sections of beads. If there are just two sections, then the necklace is 4/4 and there is only one such necklace. If there are four, then each colour is split either 2, 2 or 3, 1. So the possibilities are 2/3/2/1 (which can occur in two ways, with the 3 being one colour or the other) or 2/2/2/2 (which can occur in one way) or 3/3/1/1(also one way). Note that 3/2/1/2 appears to be another possibility, but is the same as 2/3/2/1 rotated.
If there are six sections, then each colour must be split into 2, 1, 1 and the possibilities are 2/2/1/1/1/1 (one way)or 2/1/1/2/1/1 (one way). Finally, if there are eight, then the only possible necklace is 1/1/1/1/1/1/1/1. In total that gives 8 necklaces.

2008…

6. E

6		
4	5	
1	2	3

We number the squares to identify them. The only line of symmetry possible is the diagonal through 1 and 5. For a symmetric shading, if 4 is shaded, then so too must be 2; so either both are shaded or neither. Likewise 3 and 6 go together and provide 2 more choices. Whether 1 is shaded or not will not affect a symmetry, and this gives a further 2 choices; and the same applies to 5. Overall, therefore, there are $2^4 = 16$ choices. However, one of these is the choice to shade no squares, which is excluded by the question.

2009…

6. C If at most two marbles of each colour are chosen, the maximum number we can choose is 8, corresponding to 2 of each. Therefore, if 9 are chosen, we must have at least 3 of one colour, but this statement is not true if 9 is replaced by any number less than 9.

2010…

7. C There are 24 arrangements of the letters in the word ANGLE with A as the first letter. In alphabetical order AEGLN is first and ANLGE is last ie 24th. ANLEG is the 23rd and hence ANGLE is the 22nd.

20. C Let the number of boys in the class be x. Hence $\frac{10}{10+x} \times \frac{9}{9+x} = \frac{3}{20}$.
Simplifying gives $1800 = 3(10+x)(9+x)$ and then $x^2 + 19x - 510 = 0$.
Factorising gives $(x+34)(x-15) = 0$ and, since $x \neq -34, x = 15$.

25. D The sum of 10 different digits is 45. As the sum of the digits in the question is 36 then digits adding to 9 are omitted.
The combinations of digits satisfying this are:

$$9;\ 1+8;\ 2+7;\ 3+6;\ 4+5;\ 1+2+6;\ 1+3+5;\ 2+3+4.$$

When '0' is not involved there are $(8! + 4 \times 7! + 3 \times 6!)$ numbers, whereas when '0' is used there are $(8 \times 8! + 4 \times 7 \times 7! + 3 \times 6 \times 6!)$.
This gives a total of $9 \times 8! + (4+28) \times 7! + (3+18) \times 6! = (72+32+3) \times 7! = 107 \times 7!$
Hence $N = 107$.

2012…

14. D Note that each student has a language in common with exactly four of the other five students. For instance, Jean-Pierre has a language in common with each of Ina, Karim, Lionel and Mary. Only Helga does not have a language in common with Jean-Pierre. So whichever two students are chosen, the probability that they have a language in common is 4/5.

23. C Tom wins after one attempt each if he hits the target and Geri misses. The probability of this happening is $\frac{4}{5} \times \frac{1}{3} = \frac{4}{15}$. Similarly the probability that Geri wins after one attempt is $\frac{2}{3} \times \frac{1}{5} = \frac{2}{15}$. So the probability that both competitors will have at least one more attempt is $1 - \frac{4}{15} - \frac{2}{15} = \frac{3}{5}$.
Therefore the probability that Tom wins after two attempts each is $\frac{3}{5} \times \frac{4}{15}$. The probability that neither Tom nor Geri wins after two attempts each is $\frac{3}{5} \times \frac{3}{5}$. So the probability that Tom wins after three attempts each is $\left(\frac{3}{5}\right)^2 \times \frac{4}{15}$ and, more generally, the probability that he wins after n attempts each is $\left(\frac{3}{5}\right)^{n-1} \times \frac{4}{15}$.
Therefore the probability that Tom wins is $\frac{4}{15} + \left(\frac{3}{5}\right) \times \frac{4}{15} + \left(\frac{3}{5}\right)^2 \times \frac{4}{15} + \left(\frac{3}{5}\right)^3 \times \frac{4}{15} + \ \ldots$.
This is the sum to infinity of a geometric series with first term $\frac{4}{15}$ and common ratio $\frac{3}{5}$. Its value is $\frac{4}{15} \div \left(1 - \frac{3}{5}\right) = \frac{2}{3}$.

2013…

12. B There are three options for Sammy's first choice and then two options for each subsequent choice. Therefore the number of possible ways is $3 \times 2 \times 2 \times 2 \times 2 = 48$.

2014…

15. C

A		
B	C	D
E	F	G

Label the squares as shown. Possible pairs to be shaded which include A are AD, AE, AF and AG. Pairs excluding A are BD, BG, DE, EG. Triples must include A and there are two possibilities, ADE and AEG. This gives 10 ways of shading the grid,

22. A The probability that the second marble is blue equals P(2nd marble is blue given that the 1st marble is blue) + P(2nd marble is blue given that the 1st marble is yellow), which is $\frac{m}{m+n} \times \frac{m+k}{m+n+k} + \frac{n}{m+n} \times \frac{m}{m+n+k} = \frac{m^2+mk+mn}{(m+n)(m+n+k)} = \frac{m(m+k+n)}{(m+n)(m+n+k)} = \frac{m}{m+n}$.
Note: this expression is independent of k.

2015…

16. A Let the six Fnargs in their final positions be denoted by $F_1F_2F_3F_4F_5F_6$. There are six choices for F_1. Once this Fnarg is chosen, the colours of the Fnargs must alternate all along the line and so we need only consider the number of heads. There are $3 - 1 = 2$ choices for F_2 as the number of heads for $F_2 \neq$ the number of heads for F_1. There is only one choice for F_3 as F_3 cannot have the same number of heads as F_2 or F_1 (F_3 and F_1 are the same colour and so have different numbers of heads). There is only one choice for F_4 as it is completely determined by F_3 and F_2, just as F_3 was completely determined by F_2 and F_1. There is only one choice for each of F_5 and F_6 as they are the last of each colour of Fnargs. The total number of ways of lining up the Fnargs is $6 \times 2 \times 1 \times 1 \times 1 \times 1$ which is 12.

24. C There are five cards in Peter's set that are printed with an integer that has no prime factors in common with any other number from 1 to 25. The five numbers are 1 (which has no prime factors) and the primes 13, 17, 19 and 23. These cards cannot be placed anywhere in the row of N cards. One possible row is: 11, 22, 18, 16, 12, 10, 8, 6, 4, 2, 24, 3, 9, 21, 7, 14, 20, 25, 15, 5. So the longest row is of 20 cards.

2016…

17. D One way to count the possible codes is in descending numerical order of the three-digit codes. The list begins: 789; 689, 679, 678; 589, 579, 578, 569, 568, 567; ... . Each initial digit n produces part of the list with the $(8 - n)$th triangular number of possible codes, where $n \leqslant 7$. The total number of possible codes is then the sum of these triangular numbers
$1 + 3 + 6 + 10 + 15 + 21 + 28$ including 1 code starting with the digit 7, all the way to 28 codes starting with the digit 1. The total number of codes that Aaron can choose is 84.