UKMT Indices & Surds Questions

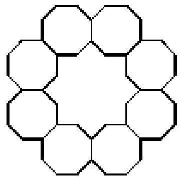
(Answers follow after all the questions)

2005…

3. What is the mean of the five numbers 1^5, 2^4, 3^3, 4^2 and 5^1 ?

A 6.2 B 11.4 C 12.2 D 13 E 13.8

17. Eight identical regular octagons are placed edge to edge in a ring in such a way that a symmetrical star shape is formed by the interior edges. If each octagon has sides of length 1, what is the area of the star?



A $5 + 10\sqrt{2}$ B $8\sqrt{2}$ C $9 + 4\sqrt{2}$ D $16 - 4\sqrt{2}$ E $8 + 4\sqrt{2}$

25. Which of the following is equal to $\dfrac{1}{\sqrt{2005 + \sqrt{2005^2 - 1}}}$?

A $\sqrt{1003} - \sqrt{1002}$ B $\sqrt{1005} - \sqrt{1004}$ C $\sqrt{1007} - \sqrt{1005}$ D $\sqrt{2005} - \sqrt{2003}$ E $\sqrt{2007} - \sqrt{2005}$

2006…

4. What is the value of $\sqrt{2^4 + \sqrt{3^4}}$?

A 4 B $\sqrt{20}$ C 5 D 7 E $\sqrt{97}$

2007…

5. Which of the five expressions shown has a different value from the other four?

A 2^8 B 4^4 C $8^{8/3}$ D 16^2 E $32^{6/5}$

2008…

3. What is the value of $\sqrt{\dfrac{1}{2^6} + \dfrac{1}{6^2}}$?

A $\dfrac{1}{10}$ B $\dfrac{1}{9}$ C $\dfrac{1}{3}$ D $\dfrac{5}{24}$ E $\dfrac{7}{24}$

13. Positive integers m and n are such that $2^m + 2^n = 1280$. What is the value of $m + n$?

A 14 B 16 C 18 D 32 E 640

2009…

3. What is the value of $1^6 - 2^5 + 3^4 - 4^3 + 5^2 - 6^1$?

A 1 B 2 C 3 D 4 E 5

11. For what value of x is $\sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} = 2^x$ true?

A $\frac{1}{2}$ B $1\frac{1}{2}$ C $2\frac{1}{2}$ D $3\frac{1}{2}$ E $4\frac{1}{2}$

2010…

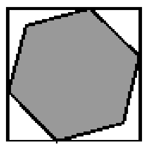
11. What is the median of the following numbers?

A $9\sqrt{2}$ B $3\sqrt{19}$ C $4\sqrt{11}$ D $5\sqrt{7}$ E $6\sqrt{5}$

17. One of the following is equal to $\sqrt{9^{16x^2}}$ for all values of x. Which one?

A 3^{4x} B 3^{4x^2} C 3^{8x^2} D 9^{4x} E 9^{8x^2}

21. The diagram shows a regular hexagon, with sides of length 1, inside a square. Two vertices of the hexagon lie on a diagonal of the square and the other four lie on the edges.



What is the area of the square?

A $2 + \sqrt{3}$ B 4 C $3 + \sqrt{2}$ D $1 + \frac{3\sqrt{3}}{2}$ E $\frac{7}{2}$

2011…

4. What is the last digit of 3^{2011} ?

A 1 B 3 C 5 D 7 E 9

2012…

4. According to one astronomer, there are one hundred thousand million galaxies in the universe, each containing one hundred thousand million stars. How many stars is that altogether?

A 10^{13} B 10^{22} C 10^{100} D 10^{120} E 10^{121}

21. Which of the following numbers does *not* have a square root in the form $x + y\sqrt{2}$, where x and y are positive integers?

A $17 + 12\sqrt{2}$ B $22 + 12\sqrt{2}$ C $38 + 12\sqrt{2}$ D $54 + 12\sqrt{2}$ E $73 + 12\sqrt{2}$

2013…

3. What is the 'tens' digit of $2013^2 - 2013$?

A 0 B 1 C 4 D 5 E 6

2014…

9. Which of the following is divisible by 9?

A $10^{2014} + 5$ B $10^{2014} + 6$ C $10^{2014} + 7$ D $10^{2014} + 8$ E $10^{2014} + 9$

23. Which of the following have no real solutions?

(i) $2x < 2^x < x^2$ (ii) $x^2 < 2x < 2^x$ (iii) $2^x < x^2 < 2x$
(iv) $x^2 < 2^x < 2x$ (v) $2^x < 2x < x^2$ (vi) $2x < x^2 < 2^x$

A (i) and (iii) B (i) and (iv) C (ii) and (iv) D (ii) and (v) E (iii) and (v)

24. Which of the following is smallest?

A $10 - 3\sqrt{11}$ B $8 - 3\sqrt{7}$ C $5 - 2\sqrt{6}$ D $9 - 4\sqrt{5}$ E $7 - 4\sqrt{3}$

2015…

1. What is $2015^2 - 2016 \times 2014$?

A -2015 B -1 C 0 D 1 E 2015

22. Let $f(x) = x + \sqrt{x^2 + 1} + \dfrac{1}{x - \sqrt{x^2 + 1}}$. What is the value of $f(2015)$?

A -1 B 0 C 1 D $\sqrt{2016}$ E 2015

2016…

7. Which of these has the smallest value?

A 2016^{-1} B $2016^{-1/2}$ C 2016^{0} D $2016^{1/2}$ E 2016^{1}

12. What is the smallest square that has 2016 as a factor?

A 42^2 B 84^2 C 168^2 D 336^2 E 2016^2

16. For which value of k is $\sqrt{2016} + \sqrt{56}$ equal to 14^k ?

A $\frac{1}{2}$ B $\frac{3}{4}$ C $\frac{5}{4}$ D $\frac{3}{2}$ E $\frac{5}{2}$

UKMT Indices & Surds Answers

2005…

3. **D** The numbers are 1, 16, 27, 16 and 5 respectively. Their sum is 65, so their mean is 13.

15. **E** Number the four statements in order from the top. If Alice is the mother, then statements 1 and 4 are both true. If Beth is the mother, then statements 2 and 3 are both true. If Carol is the mother, then all four statements are false. If Diane is the mother, then statements 2 and 4 are both true. However, if Ella is the mother then statements 1, 2 and 3 are false and statement 4 is true, as required.

25. **A**

$$\frac{1}{\sqrt{2005+\sqrt{2005^2-1}}} = \frac{1}{\sqrt{1003+1002+\sqrt{(2005+1)(2005-1)}}}$$

$$= \frac{1}{\sqrt{(\sqrt{1003})^2+2\sqrt{1003}\sqrt{1002}+(\sqrt{1002})^2}} = \frac{1}{\sqrt{(\sqrt{1003}+\sqrt{1002})^2}} = \frac{1}{\sqrt{1003}+\sqrt{1002}}$$

$$= \frac{(\sqrt{1003}-\sqrt{1002})}{(\sqrt{1003}-\sqrt{1002})(\sqrt{1003}+\sqrt{1002})} = \frac{(\sqrt{1003}-\sqrt{1002})}{1003-1002} = \sqrt{1003}-\sqrt{1002}.$$

2006…

4. **C** $\sqrt{2^4+\sqrt{3^4}} = \sqrt{16+\sqrt{81}} = \sqrt{16+9} = \sqrt{25} = 5.$

2007…

5. **E** $4^4 = (2^2)^4 = 2^8$; $8^{8/3} = (2^3)^{8/3} = 2^8$; $16^2 = (2^4)^2 = 2^8$. However, $32^{6/5} = (2^5)^{6/5} = 2^6$.

2008…

3. **D** $\frac{1}{2^6}+\frac{1}{6^2} = \frac{3^2+2^4}{2^6\times 3^2} = \frac{25}{2^6\times 3^2} = \frac{5^2}{(2^3\times 3)^2}$. Hence the answer is $\frac{5}{2^3\times 3} = \frac{5}{24}$.

13. **C** Since $1280 = 2^8\times 5 = 2^8(2^0+2^2) = 2^8+2^{10}$, we may take $m=8$ and $n = 10$ (or vice versa) to get $m+n = 8+10 = 18$. It is easy to check that there are no other possibilities.

2009…

3. **E** $1 - 32 + 81 - 64 + 25 - 6 = 5.$

11. **C** $\sqrt{2}+\sqrt{2}+\sqrt{2}+\sqrt{2} = 4\sqrt{2} = 2^2\times 2^{1/2} = 2^{2\frac{1}{2}}$. Hence $x = 2\frac{1}{2}$.

2010…

11. D Squaring the numbers given allows us to see their order easily:

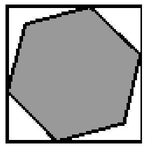
$(9\sqrt{2})^2 = 81 \times 2 = 162$ $\quad$ $(3\sqrt{19})^2 = 9 \times 19 = 171$ $\quad$ $(4\sqrt{11})^2 = 16 \times 11 = 176$

$(5\sqrt{7})^2 = 25 \times 7 = 175$ $\quad$ $(6\sqrt{5})^2 = 36 \times 5 = 180$

As 175 is the middle one of these numbers, the answer is $5\sqrt{7}$.

17. E $\sqrt{9^{16x^2}} = 9^{(16x^2)/2} = 9^{8x^2}$.

21. A The hypotenuse of one of the small right-angled triangles is parallel to the diagonal and hence makes angles of 45°. Since the hypotenuse has length 1, the other two sides have length $\frac{1}{\sqrt{2}}$, by Pythagoras' Theorem.



As the internal angle of a regular hexagon is 120°, drawing a diagonal from NW to SE forms two triangles, bottom right, each with angles 45°, 120° and 15°. (The sum of the angles in a triangle is 180°).

Let the square have length y units. Using the Sine Rule gives $\dfrac{y - \frac{1}{\sqrt{2}}}{\sin 120°} = \dfrac{1}{\sin 45°}$.

Hence $y - \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{3}}{2} \div \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{3}}{\sqrt{2}}$ and therefore $y = \dfrac{\sqrt{3} + 1}{\sqrt{2}}$.

Hence the area of the square is $y^2 = \left(\dfrac{\sqrt{3} + 1}{\sqrt{2}}\right)^2 = \dfrac{4 + 2\sqrt{3}}{2} = 2 + \sqrt{3}$.

2011…

4. D Since $3^1 = 3$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$, $3^5 = 243$, … we see that the final digits cycle through the four numbers 3, 9, 7, 1. As $2011 = 502 \times 4 + 3$, the last digit of 3^{2011} is 7.

2012…

4. B One hundred thousand million is $10^2 \times 10^3 \times 10^6 = 10^{11}$. So the number of stars is $10^{11} \times 10^{11} = 10^{22}$.

21. D $(x + y\sqrt{2})^2 = x^2 + 2xy\sqrt{2} + 2y^2$. Note that all of the alternatives given are of the form $a + 12\sqrt{2}$ so we need $xy = 6$. The only ordered pairs (x, y) of positive integers which satisfy this are (1, 6), (2, 3), (3, 2), (6, 1). For these, the values of $x^2 + 2y^2$ are 73, 22, 17, 38 respectively. So the required number is $54 + 12\sqrt{2}$.

2013…

3. D Factorising $2013^2 - 2013$ gives $2013(2013 - 1)$ which equals 2013×2012. So the tens digit is 5 as $13 \times 12 = 156$ and only this part of the product contributes to the tens digit of the answer.

2014…

9. **D** A number is divisible by 9 if and only if its digit sum is divisible by 9. The number 10^{2014} can be written as a 1 followed by 2014 zeros, so this part has a digit sum of 1. Of all the options given, only adding on 8 to this will make a digit sum of 9, so $10^{2014} + 8$ is the required answer.

23. **E** If the graphs of $y = 2x$, $y = 2^x$ and $y = x^2$ are sketched on the same axes it can be seen that case (i) holds for $2 < x < 4$, case (ii) holds for $0 < x < 1$, case (iv) holds for $1 < x < 2$ and case (vi) holds for $x > 4$.
There are no real solutions for case (iii). Consider $x^2 < 2x$, which is true for $0 < x < 2$. However for $0 < x < 2$ it can be seen that $2^x > x^2$ rather than $2^x < x^2$ as stated.
There are no real solutions for case (v). Consider $2x < x^2$, which is true for $x < 0$ or $x > 2$. However, when $x < 0$ we have $2^x > 2x$ as 2^x is positive and $2x$ is negative, rather than $2^x < 2x$ as stated. Also, when $x = 2$ we have $2^x = 2x$, but for $x > 2$, $2^x > 2x$ rather than $2^x < 2x$ as stated.

24. **A** Each of the five expressions can be written in the form $\sqrt{x} - \sqrt{x-1}$, where x is in turn 100, 64, 25, 81 and 49. As $(\sqrt{x} - \sqrt{x-1})(\sqrt{x} + \sqrt{x-1}) = x - (x-1) = 1$, we can write $\left(\sqrt{x} - \sqrt{x-1}\right) = \dfrac{1}{(\sqrt{x} + \sqrt{x-1})}$. Since $(\sqrt{x} + \sqrt{x-1})$ increases with x, then $(\sqrt{x} - \sqrt{x-1})$ must decrease with x. Therefore, of the given expressions, the one corresponding to the largest value of x is the smallest. This is $\sqrt{100} - \sqrt{99}$ which is $10 - 3\sqrt{11}$.

2015…

1. **D** The expression $2015^2 - 2016 \times 2014$ can be written as $2015^2 - (2015+1)(2015-1)$ which simplifies, using the difference of two squares, to $2015^2 - (2015^2 - 1) = 1$.

22. **B** $f(x) = x + \sqrt{x^2+1} + \dfrac{1}{x - \sqrt{x^2+1}} = \dfrac{(x + \sqrt{x^2+1})(x - \sqrt{x^2+1}) + 1}{x - \sqrt{x^2+1}}$. The numerator is $x^2 - \left(\sqrt{x^2+1}\right)^2 + 1 = -1 + 1 = 0$. So $f(x) = 0$. Hence $f(2015) = 0$.

2016…

7. **A** The number 2016^0 has value 1. As $2016 > 1$, $2016^{1/2} < 2016^1$. The values of their reciprocals, $2016^{-1/2}$ and 2016^{-1} are then in the opposite order. So the five options given are in numerical order, with 2016^{-1}, or $\frac{1}{2016}$, being the smallest.

12. **C** The prime factorisation of 2016 is $2^5 \times 3^2 \times 7$. To create the smallest square which is a multiple of 2016, the powers of each prime must be as small as possible and even, whilst also being at least as big as those in the prime factorisation of 2016. This gives $2^6 \times 3^2 \times 7^2$ which is $(2^3 \times 3 \times 7)^2$ or 168^2.

16. **D** The expression $\sqrt{2016} + \sqrt{56}$ can be written as $\sqrt{2^5 \times 3^2 \times 7} + \sqrt{2^3 \times 7}$ which is $\sqrt{4^2 \times 3^2 \times 2 \times 7} + \sqrt{2^2 \times 2 \times 7}$. This simplifies to $12\sqrt{14} + 2\sqrt{14}$ which is $14\sqrt{14}$ and, using index notation, this can be written as $14^{3/2}$. Hence $k = \frac{3}{2}$.